Longitudinal dynamics

Electrostatic accelerators

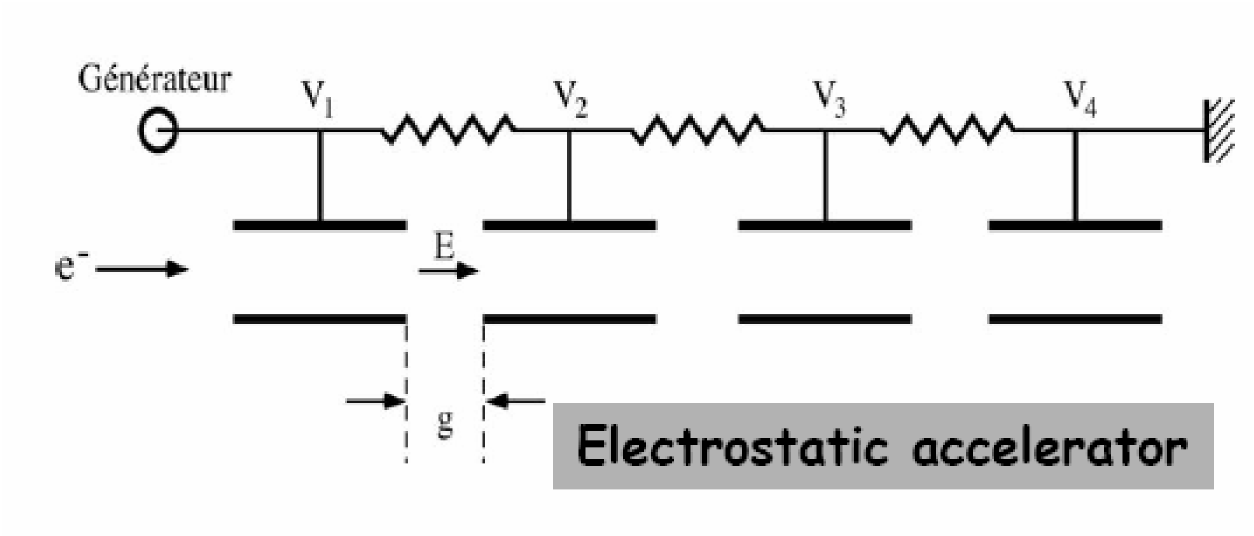
- Particles are accelerated by a constant voltage across a gap
- This acceleration is limited by breakdown voltages even in the tandem or Van der Graff accelerators

Energy gain:

$$W = nq\sum V_n$$

Limited by the generator

$$V_{generator} = \sum V_n$$



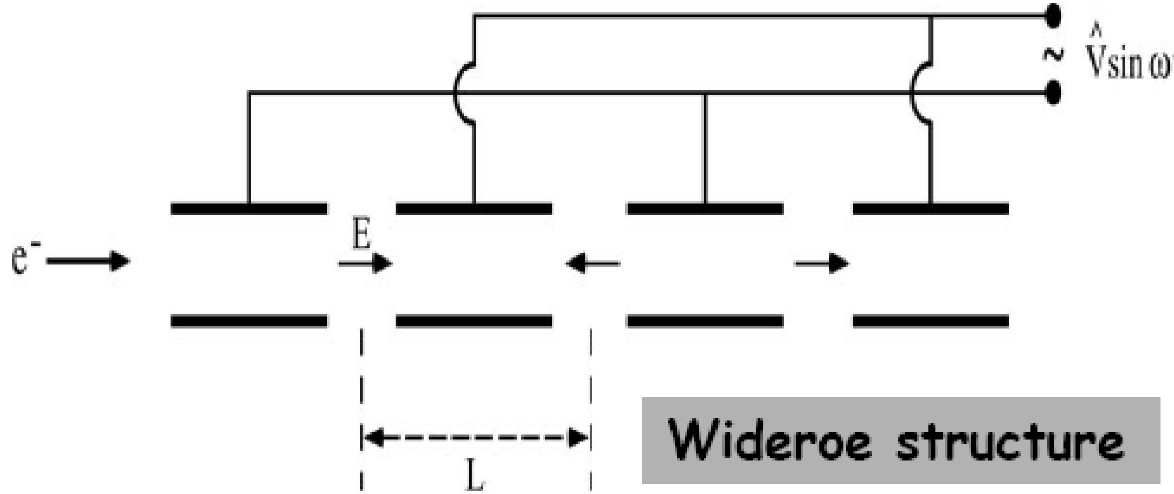
RF accelerators

- The use of RF fields allows an arbitrary number of accelerating steps. The electric field is not longer continuous but sinusoidal alternating half periods of acceleration and deceleration.

$$L = vT/2 = \beta c \frac{\pi}{\omega_{RF}} = \beta\lambda/2$$

T RF period

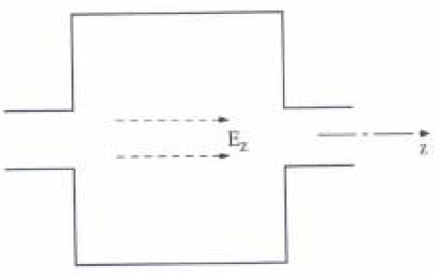
V particle velocity



Particles need to be grouped!!

Energy gain in the RF case

The energy gain of a particle of charge q traversing the gap is



$$W(r,t) = q\int E_z(r,z,t)dz$$

with a constant amplitude field E_0

$$E_z(r,z,t) = E_0 \cos(\omega_{RF} t + \phi)$$

Assume the particle is synchronized and is at the center of the gap when $\phi = 0$

$$\Delta W = qE_0 \int_{-q/2}^{g/2} \cos(\omega_{RF} \frac{z}{v} dz)$$

with z = vt

$$\Delta W = qV \frac{\sin \Theta / 2}{\Theta / 2} = qV\,\mathrm{T}$$

Transit time factor

$$T = \frac{\sin(\omega g / 2v)}{\omega g / 2v}$$

- T is known as the transit time factor, in general:

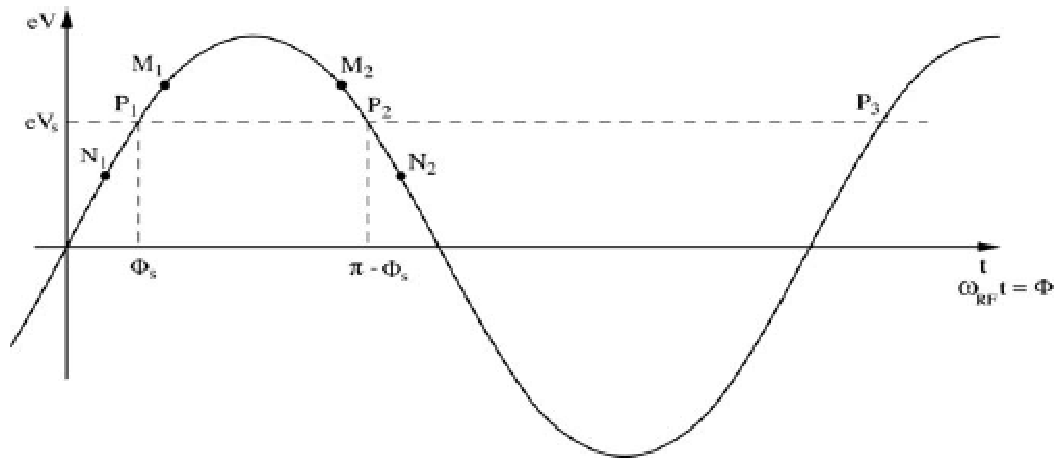
$$\mathrm{T} = \frac{\int_{-g/2}^{g/2} E(0,z)\cos\omega t(z)dz}{\int_{-g/2}^{g/2} E(0,z)dz}$$

- The energy gain is a function of the particle velocity
- The energy gain of a particle with velocity v / maximum energy gain (for a particle with v → ∞)
- It can be shown that T does not depend on the instant on which the particle crosses the gap
- The shorter the gap length g with respect to the distance the beam travels in one period, the higher the transit time factor

Phase stability

In general, there is two phases for which the velocity gain is nominal.

Around point P_1, particles that arrive earlier (N_1) experience a smaller accelerating field



Particles arriving later (N_2) will be accelerated more

A restoring force that keeps particles oscillating around a stable phase called the synchronous phase ϕ_s

What will happen if we chose $\phi_s = 0$?

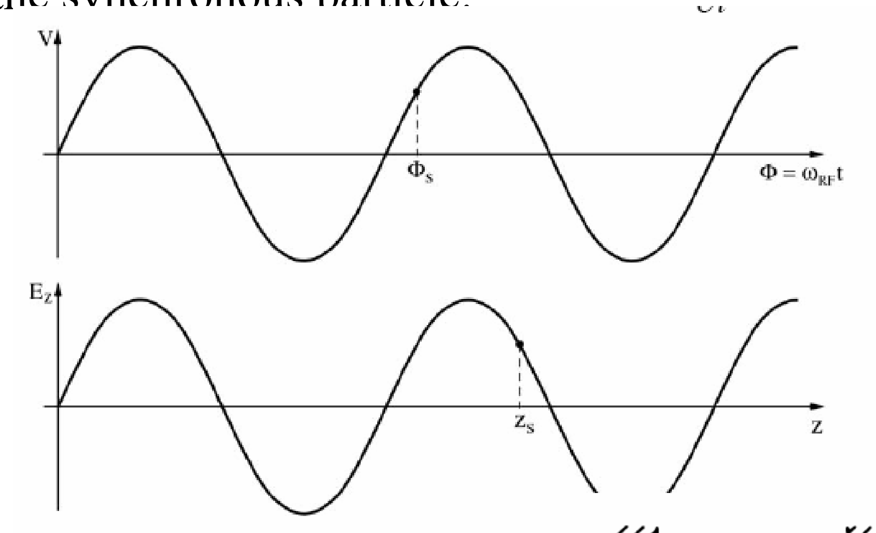
RF de-focusing

A consequence of phase stability is what is called RF de-focusing.

We just learnt that to have stability, a longitudinal focusing is required.

In the reference frame moving with the synchronous particle:

$$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E}{\partial z} < 0$$

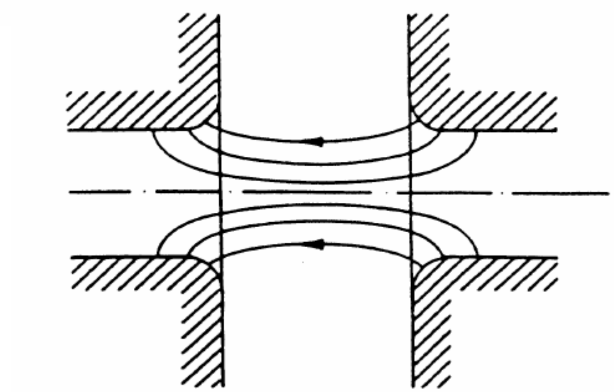


In the absence of electric charge

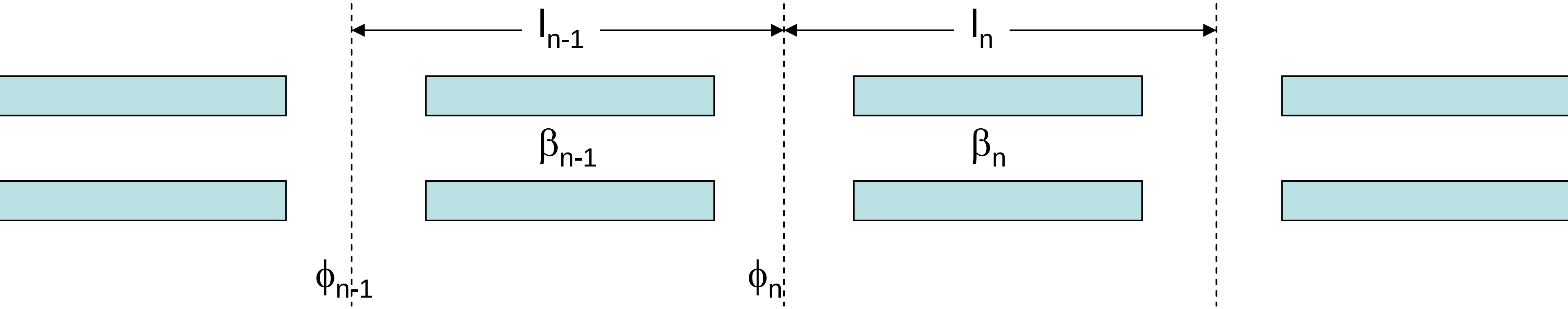
$$\nabla \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} \neq 0$$

where x represents the generic transverse direction.

<u>External focusing is required!!</u>



Equations of motion I



An array of accelerating cells with varying length matching $l_n=\beta_{s,n}\lambda$

For an arbitrary particle, the new phase will be:

$$\phi_n = \phi_{n-1} + \omega \frac{l_{n-1}}{\beta_{n-1} c} = \phi_{n-1} + 2\pi \frac{\beta_{s,n-1}}{\beta_{n-1}}$$

We will use the relative variable $\phi-\phi_s$

$$\Delta(\phi-\phi_s)_n = \Delta\phi_n - \Delta\phi_{s,n} = 2\pi\beta_{s,n-1}\left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}}\right] \cong -2\pi\beta_{s,n-1}\frac{\delta\beta_{n-1}}{\beta_{s,n-1}{}^2}$$

where

$$\frac{1}{\beta} - \frac{1}{\beta_s} = \frac{1}{\beta_s+\delta\beta} - \frac{1}{\beta_s} \cong -\frac{\delta\beta}{\beta_s{}^2}, \text{ for } \delta\beta << 1$$

Equations of motion II

Instead of using β we use the kinetic energy W

$$\Delta(\phi-\phi_s)_n = -2\pi \frac{\left(W_{n-1} - W_{s,n-1}\right)}{mc^2 \gamma^3{}_{s,n-1} \beta_{s,n-1}{}^2}$$

Now, we derive the difference equation for the kinetic energy gain

$$\Delta W_n = W_n - W_{n-1} = qE_0 T l_n \cos\phi_n$$

$$\Delta\left(W - W_n\right) = qE_0 T l_n (\cos\phi_n - \cos\phi_{n,s})$$

The two equations are coupled. Assuming slow acceleration or small ϕ_s and after some manipulation we obtain the following set of differential equations.

$$\frac{dw}{ds} = \frac{qE_0 T}{mc_2}\left(\cos\phi - \cos\phi_s\right) \quad \text{and} \quad \frac{d\phi}{ds} = -\frac{2\pi}{\beta_s{}^3 \gamma_s{}^3 \lambda} w$$

with $w = \dfrac{W - W_s}{mc^2}$

Longitudinal oscillation

The two equations are coupled and it is easy to see that the motion will be an oscillation.

$$\frac{d^2\phi}{ds^2} = -\frac{2\pi q E_0 T}{mc^2 \beta_s{}^3 \gamma_s{}^3 \lambda}\left(\cos\phi - \cos\phi_s\right)$$

The exact solutions are calculated numerically. In the approximation of small oscillations:

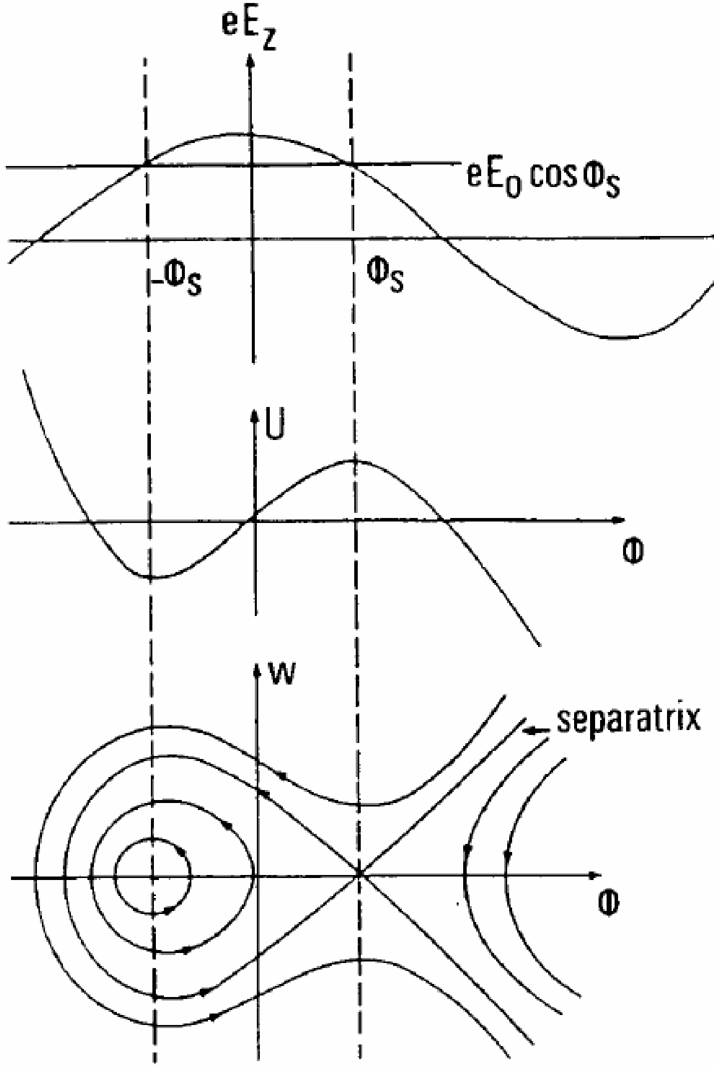
$$\frac{d^2\varphi}{ds^2} + \frac{2\pi q E_0 T \sin(-\phi_s)}{mc^2 \beta_s{}^3 \gamma_s{}^3 \lambda}\varphi = 0$$

with $\varphi = \phi - \phi_s$

$$\Omega^2 = \frac{2\pi q E_0 T \sin(-\phi_s)}{mc^2 \beta_s{}^3 \gamma_s{}^3 \lambda}$$

is the angular frequency of these small oscillations. For stability $\Omega_s > 0$ and $\sin\phi_s < 0$ as required for longitudinal focusing (?)

The separatrix



The accelerating field is a function of the synchronous phase

The stable area is called "bucket" and is also referred to as "the golf club"

Particles will follow a trajectory depending on their initial conditions

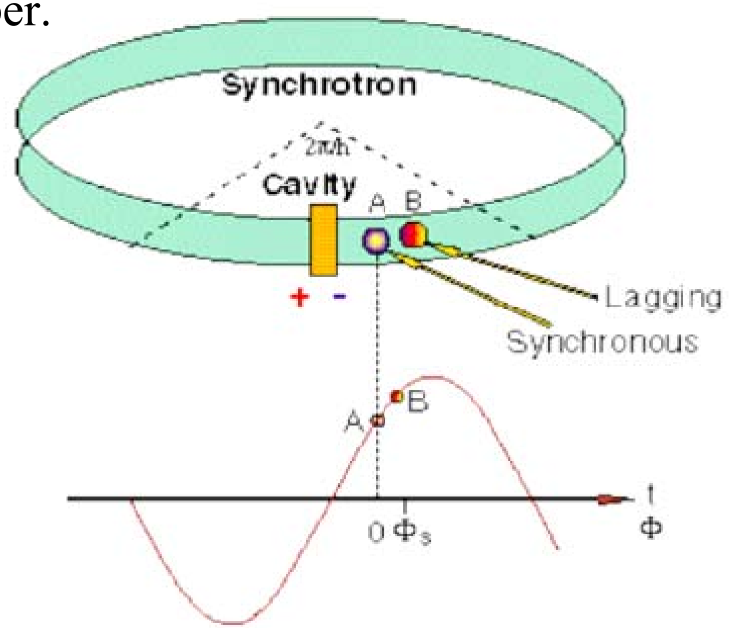
Slip factor

Up to know we have only treated linear accelerators.

For cyclotrons, the trajectory of particles increases naturally and the synchronism is not guaranteed

In a synchrotron, *l* is the circumference of the accelerator and thus fixed. The magnetic field is ramped in synchronism with the energy of the beam so as to keep the synchronous phase. The RF frequency is an integer number of the revolution frequency $f_{RF} = h\phi_{rev}$ known as the harmonic number.

However the kinetic energy difference among particles in the bunch causes orbit length differences



Momentum compaction

$$\alpha = \frac{p}{R}\frac{dR}{dp} = \frac{\langle D_x \rangle_m}{R} \cong 1/Q_x^{\ 2}$$

Slip factor

$$\eta = \frac{p}{f_{rev}}\frac{df_{rev}}{dp} = \frac{1}{\gamma^2} - \alpha$$

Transition

Particles having a higher kinetic energy may then arrive later than the synchronous particle (?!!). Longitudinal stability is not longer valid and in fact the synchronous stable phase is now $-\phi_s$!

The transition energy γ_{tr} is the energy which corresponds to

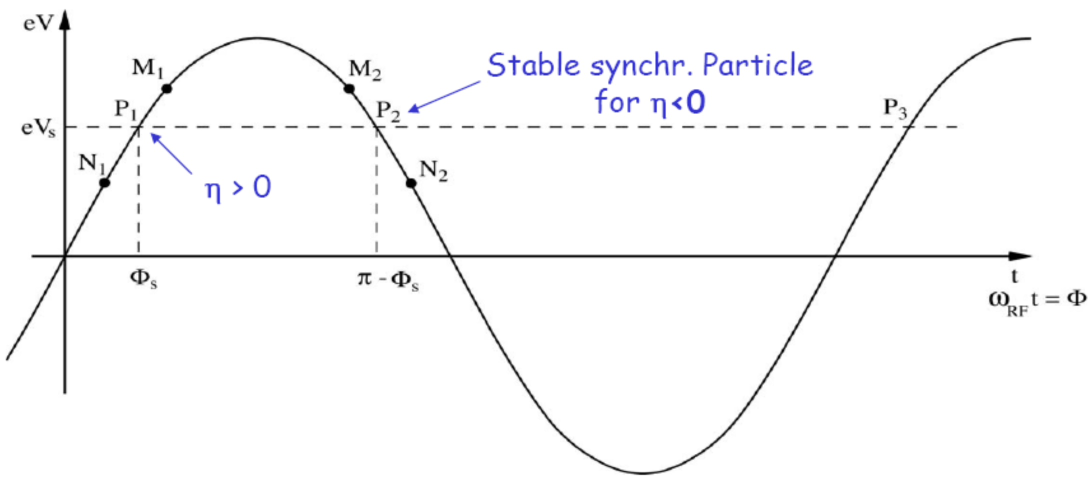
$$\frac{1}{\gamma_{tr}^{\;2}} = \alpha \quad \text{and} \quad \gamma_{tr} = \sqrt{1/\alpha} \cong Q_x$$

At this energy, all particles see the same circumference and the longitudinal oscillation is frozen.

A gamma jump has to be done to avoid all sorts of negative effects in the beam

Most synchrotrons operate above transition.

Phase stability above transition



- A new set of equations for synchrotrons that take into account the slip factor.

$$\frac{dW}{ds} = \frac{q\hat{V}}{2\pi h}\left(\sin\phi - \sin\phi_s\right) \qquad \text{and} \qquad \frac{d\phi}{dt} = \frac{h^2\omega^2{}_{rev}\eta(W)}{E_s\beta_s{}^2}W$$

$$\text{with} \quad W = \frac{\Delta E}{h\omega_s}$$