Off-momentum dynamics

Y. Papaphilippou, N. Catalan Lasheras

USPAS, Cornell University, Ithaca, NY

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Outline

- Off-momentum particles
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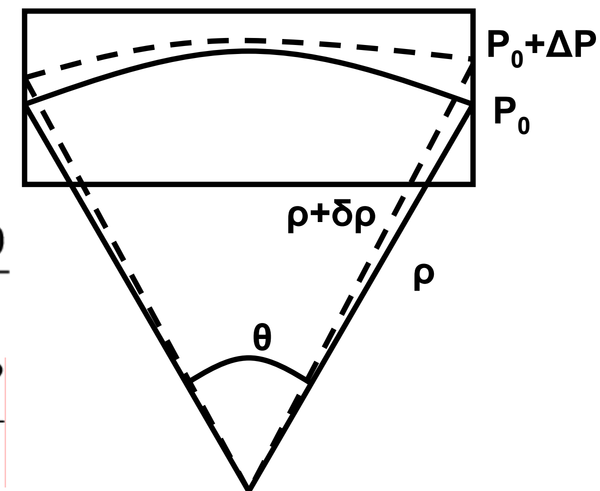
Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum $\mathbf{P_0}$
- What happens for off-momentum particles, i.e. particles with momentum $\mathbf{P_0+\Delta P}$?
- Consider a dipole with field **B** and bending radius **ρ**
- Recall that the magnetic rigidity is $B\rho = \frac{P_0}{q}$ and for off-momentum particles



$$B(\rho+\Delta\rho) = \frac{P_0+\Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Considering the effective length of the dipole unchanged

$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Off-momentum particles get different deflection (different orbit)

$$\Delta\theta = -\theta\frac{\Delta P}{P_0}$$

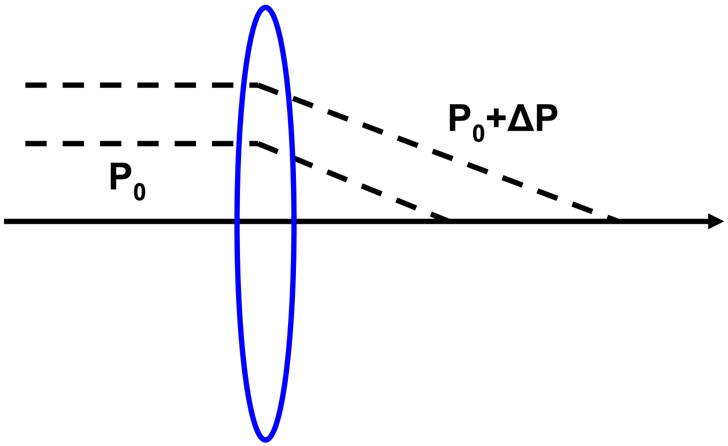
Off-momentum particles and quadrupoles

- Consider a quadrupole with gradient **G**
- Recall that the normalized gradient is

$$K = \frac{q\,G}{P_0}$$

and for off-momentum particles

$$\Delta K = \frac{dK}{dP}\Delta P = -\frac{qG}{P_0}\frac{\Delta P}{P_0}$$



- Off-momentum particle gets different focusing

$$\Delta K = -K\frac{\Delta P}{P_0}$$

- This is equivalent to optical lenses and light of different wavelengths
- In an accelerator, it means that off-momentum particles will have different optics functions depending on momentum spread

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the homogeneous equation (on-momentum) and the inhomogeneous (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way we split the equations of motion in two parts

$$\begin{aligned} x_H'' + K_x(s)x_H &= 0 \\ x_I'' + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- We may define the **dispersion function** $D(s) = \frac{x_I(s)}{\Delta P/P}$
- The dispersion equation is

$$D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}$$

Dispersion solution for a bend

- Simple solution by considering motion through a sector dipole with constant bending radius **ρ**
- The dispersion equation becomes $D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$
- The solution of the homogeneous is harmonic with frequency **1/ρ**
- A particular solution for the inhomogeneous is $D_p = \text{constant}$ and we get by replacing $D_p = \rho$
- Setting **D(0) = D$_0$** and **D'(0) = D'$_0$**, the solutions for dispersion are

$$\begin{aligned} D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0' \rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\ D'(s) &= -\frac{D_0}{\rho} \sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho}) \end{aligned}$$

General dispersion solution

- General solution possible with perturbation theory and use of Green functions
- For a general matrix $\mathcal{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S(s)' \end{pmatrix}$ the solution is

$$D(s) = S(s) \int_{s_0}^{s} \frac{C(\bar{s})}{\rho(\bar{s})} d\bar{s} + C(s) \int_{s_0}^{s} \frac{S(\bar{s})}{\rho(\bar{s})} d\bar{s}$$

- One can verify that this solution indeed verifies the differential equation of the dispersion (and the sector bend)
- The general betatron solutions can be obtained by 3X3 transfer matrices including dispersion

$$\mathcal{M}_{3\times3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

- Recalling that $x(s) = x_B(s) + D(s)\frac{\Delta P}{P}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \mathcal{M}_{3\times3} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

- For **drifts** and **quadrupoles** which do not create dispersion the 3x3 transfer matrices are just

$$\mathcal{M}_{\text{drift,quad}} = \begin{pmatrix} \mathcal{M}_{2\times 2} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- For the deflecting plane of a **sector bend** we have seen that the matrix is

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and in the non-deflecting plane is just a drift.

3x3 transfer matrices - Synchrotron magnet

- Synchrotron magnets have focusing and bending included in their body. From the solution of the sector bend, by replacing **1/ρ** with **K=(1/ρ² - k)**$^{1/2}$.

- For **K>0**

$$\mathcal{M}_{\rm syF} = \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sqrt{K}} & \frac{1-\cos\psi}{\rho K} \\ -\sqrt{K}\sin\psi & \cos\psi & \frac{\sin\psi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$$

- For **K<0**

$$\mathcal{M}_{\rm syF} = \begin{pmatrix} \cosh\psi & \frac{\sinh\psi}{\sqrt{|K|}} & -\frac{1-\cosh\psi}{\rho|K|} \\ \sqrt{|K|}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$\psi = \sqrt{|k + \frac{1}{\rho^2}|s}$$

3x3 transfer matrices - Rectangular magnet

- The end field of a rectangular magnet is simply the one of a quadrupole. The transfer matrix for the edges is

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan(\theta/2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

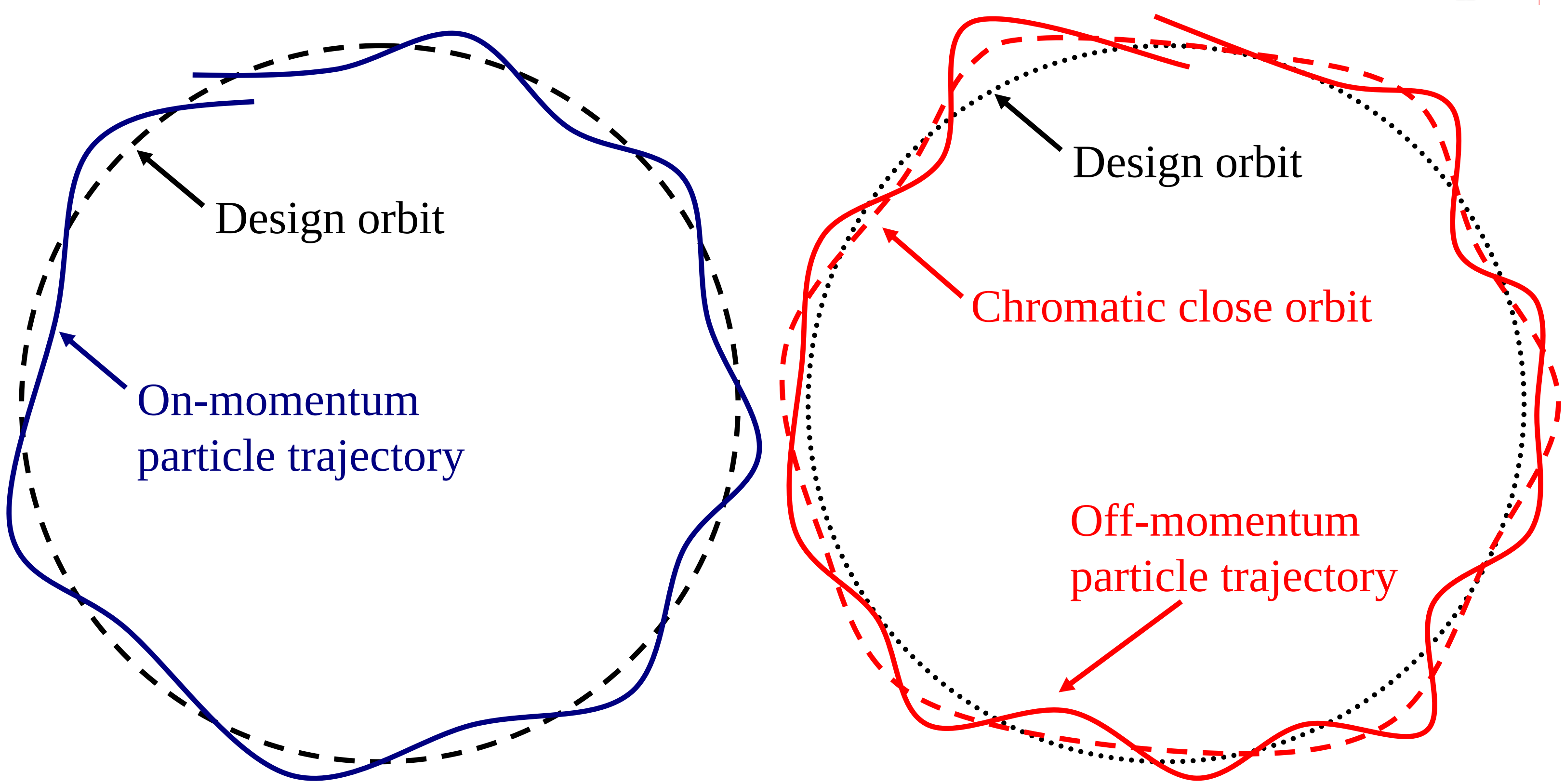
- The transfer matrix for the body of the magnet is like the sector
- The total transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sect}} \cdot \mathcal{M}_{\text{edge}}$

$$\mathcal{M}_{\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan(\theta/2) \\ 0 & 0 & 1 \end{pmatrix}$$

Chromatic closed orbit

- Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit
- Distance from the design orbit depends linearly with momentum spread and dispersion

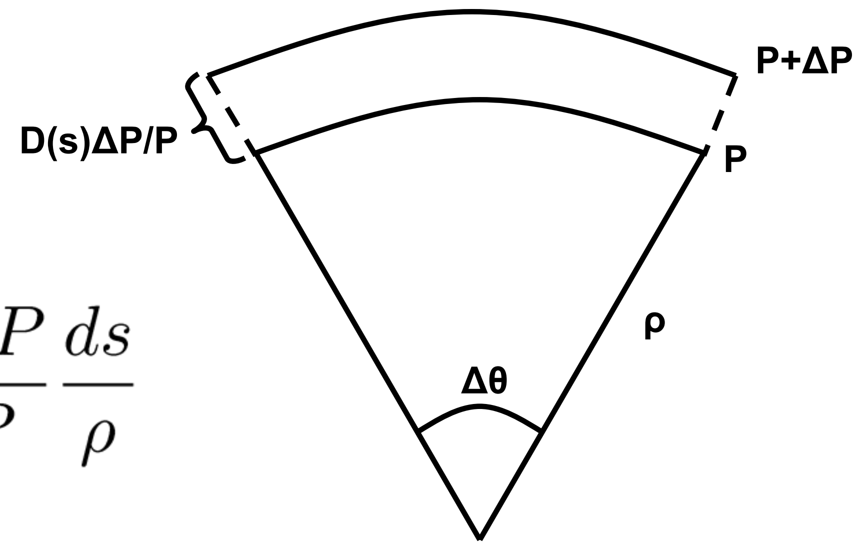
$$x_D = D(s)\frac{\Delta P}{P}$$



Momentum compaction

- Off-momentum particles on the dispersion orbit travel in a different path length than on-momentum particles
- The change of the path length with respect to the momentum spread is called **momentum compaction**

$$\alpha_c = \frac{\Delta C}{C} / \frac{\Delta P}{P}$$



- The change of circumference is

$$\Delta C = \oint D \frac{\Delta P}{P} d\theta = \oint D \frac{\Delta P}{P} \frac{ds}{\rho}$$

- So the momentum compaction is

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds = \left\langle \frac{D(s)}{\rho(s)} \right\rangle$$

Transition energy

- The revolution frequency of a particle is $f = \frac{v}{2\pi\rho} = \frac{\beta c}{2\pi\rho}$
- The change in frequency is $\frac{\Delta f}{f} = \frac{\Delta\rho}{\rho} - \frac{\Delta\beta}{\beta}$
- From the relativisitc momentum $Pc = \beta E$ we have

$$\frac{\Delta P}{P} = \frac{\Delta\beta}{\beta} + \frac{\Delta E}{E} \rightarrow \beta^2 \frac{\Delta P}{P}$$

for which $\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta P}{P}$ and the revolution frequency

$\frac{\Delta f}{f} = (\frac{1}{\gamma} - \alpha_c)\frac{\Delta P}{P}$. The transition energy is defined by

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$