RF Cavities

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How to produce an RF field

- In free space, the electromagnetic wave has electric and magnetic field vectors perpendicular to the direction of propagation.
- In bounded media, where the boundary is a perfect conductor, such a wave type is not possible because the boundary conditions cannot be satisfied.
- On the boundary, the tangential component of the electric field E_t, as well as the normal component of the magnetic field B_n, have to be zero.
- These conditions are satisfied if the wave has one of the field components in the direction of propagation.
  - If it is the electric field component E_z, the wave type is called **transverse magnetic** (TM);
  - if it is the magnetic field component B_z the wave type is called **transverse electric** (TE).
- In rectangular cavities the index indicate the number of half waves in the x and y direction, respectively. A third subscript indicates the number of half waves in the z direction. TE_{mnl} or TM_{mnl}
- We use electromagnetic waves in properly bounded media to accelerate the beam. They need t be synchronized with the passage of the particle.

Reminder: From Maxwell equations we obtained that in a medium without current or charges the propagation of an electromagnetic wave is described then by the general wave equation

$$\nabla^2 E = \nabla \times \nabla \times E = \nabla \times \left(-\frac{\partial}{\partial t} B \right) = -\frac{\partial}{\partial t}(\nabla \times B) = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

We separate the time and spatial dependence $E(\vec{r},t) = E(\vec{r})T(t) = E(\vec{r})e^{i\omega t}$

and we obtain:

$$\nabla^2 E(\vec{r}) + k^2 E(\vec{r}) = 0 \quad \text{with} \quad k = \frac{\omega}{c}$$

Because we want a net acceleration, we need an electrical field in the direction of movement. We consider only the z-component of the field.

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -k^2 E_z$$

Using a trial solution $E_z(x,y,z)=X(x)\cdot Y(y)\cdot Z(z)$ gives:

$$\frac{\partial^2 X / \partial x^2}{X} + \frac{\partial^2 Y / \partial y^2}{Y} + \frac{\partial^2 Z / \partial z^2}{Z} = -k^2$$

For this equation to hold for al values

$$\frac{\partial^2 X/\partial x^2}{X} = -k_x{}^2; \quad \frac{\partial^2 Y/\partial y^2}{Y} = -k_y{}^2; \quad \frac{\partial^2 Z/\partial z^2}{Z} = -k_z{}^2$$

$$\text{with} \quad k_x{}^2 + k_y{}^2 + k_z{}^2 = k^2$$

k_x, k_y, k_z are called the wavenumbers and they are coupled

In particular, the solution for the electric field in the propagating direction will be a wave

$$E_z(x,y,z,t) = E_{z0} X(x) Y(y) e^{i(\omega t - k_z z)} \quad \text{with} \quad k_z = \sqrt{k^2 - (k_x^2 + k_y^2)}$$

If k_z is complex the amplitude falls exponentially (damping)

If k_z is real the wave propagates

$k_c = \sqrt{(k_x{}^2 + k_y{}^2)}$ is the wavenumber of the first propagating wave and is called the cut-off wavenumber

The functions X(x)and Y(y) will be defied by the boundaries

Cylindrical waveguides

Cylindrical waveguides are used as accelerating structures.

In cylindrical coordinates, the wave equation has the following expression:

$$\frac{\partial E_z}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}-\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}=0$$

Separating now E(r)=R(r)Φ(ϕ)Z(z)T(t),The solutions for Z(z) and T(t) are similar as in the rectangular case. $\Phi(\phi)=e^{-in\phi}$. For the radial function dependence, we obtain

$$\frac{\partial^2 E_z}{\partial r^2}+\frac{1}{r}\frac{\partial E_z}{\partial r}+\left(\underbrace{\frac{\omega^2}{c^2}-k_z^{\ 2}}_{k_c}-\frac{m^2}{r^2}\right)E_z=0$$

The solutions are given by the Bessel functions of order n:

$$E_z=E_0 J_n\left(k_c r\right)e^{i(\omega t-n\phi-kz)}$$

We will place the metallic boundaries at the zeros of the Bessel functions

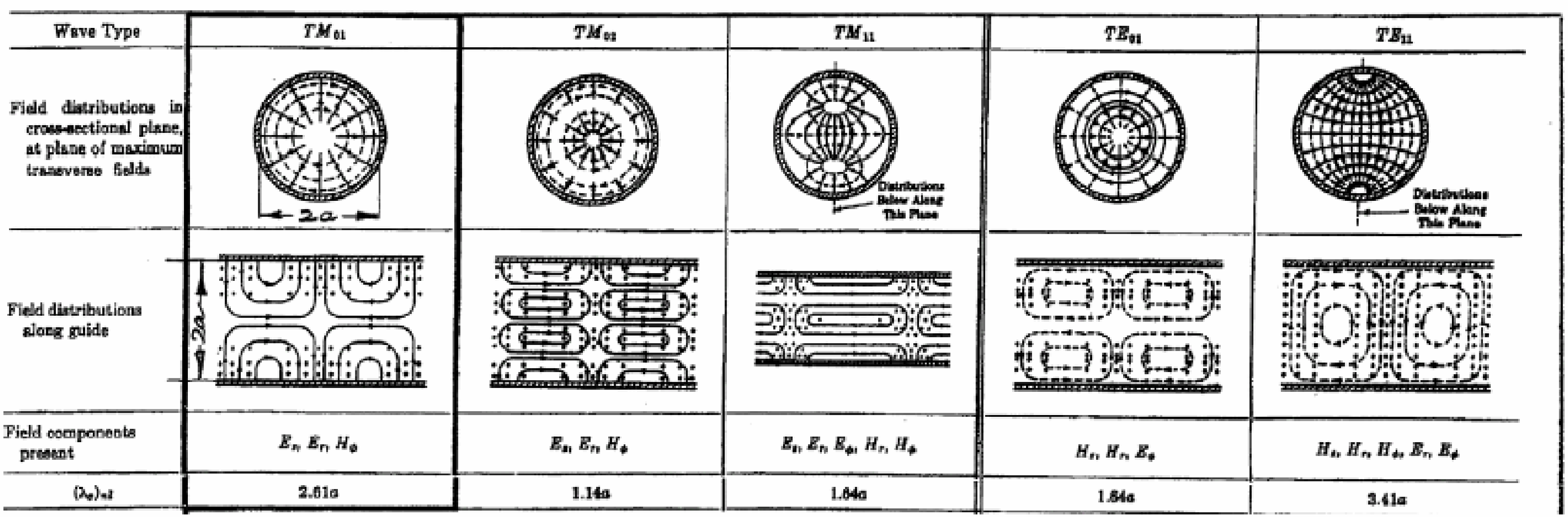
Modes in cylindrical waveguides

E_r, E_θ, H_r and H_θ can be derived from this

$$E_r = -iE_0 \frac{k_z}{k_c} J'_n(k_c r) e^{i(\omega t - kz - n\theta)} \quad ; \quad Br = -\frac{nk}{k_c{}^2} E_0 \frac{1}{r} J_n(k_c r) e^{i(\omega t - kz - n\theta)}$$

$$E_\theta = -\frac{k_z n}{k_c 2} E_0 \frac{1}{r} J_n(k_c r) e^{i(\omega t - kz - n\theta)} \quad ; \quad B_\theta = -i\frac{k}{k_c} E_0 J'_0 (k_c r) e^{i(\omega t - kz)}$$

$$E_z = E_0 J_n(k_c r) e^{i(\omega t - kz - n\theta)} \quad ; \quad B_z = 0$$



Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$
$(\lambda_c)_{cl}$	2.61a	1.14a	1.64a	1.64a	3.41a

Cylindrical cavities

The speed of the E_z crest is called the phase velocity

$$v_{ph} = \frac{z}{t} = \frac{\omega}{k_z} > \frac{\omega}{k} = c \qquad k_z = \sqrt{k^2 - k_c^2}$$

The phase velocity is above the velocity of light. If we want synchronism we need to slow down the phase velocity!!

We can do that by adding endcaps in the waveguide. The wave will be reflected and we will have a standing wave

That adds another boundary and now

$$ik_z = \frac{p\pi}{l}$$

Where l is the size of the cavity in the longitudinal direction. TM modes have now three indexes indicating the number of zeros in each direction r, ϕ, and z

The cut-off wavenuber will now be:

$$k_c^2 = \frac{\omega^2}{c^2} - \frac{p^2\pi^2}{l^2}$$

The pill box cavity

The simplest cavity consists of a cylindrical waveguide of length *l* and radius a closed at both ends

The boundary conditions are

$$E_r = E_\theta = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = l$$

$$E_z = E_\theta = 0 \quad \text{for} \quad r = a$$

The simplest solution is the mode TM_{010} with only three components

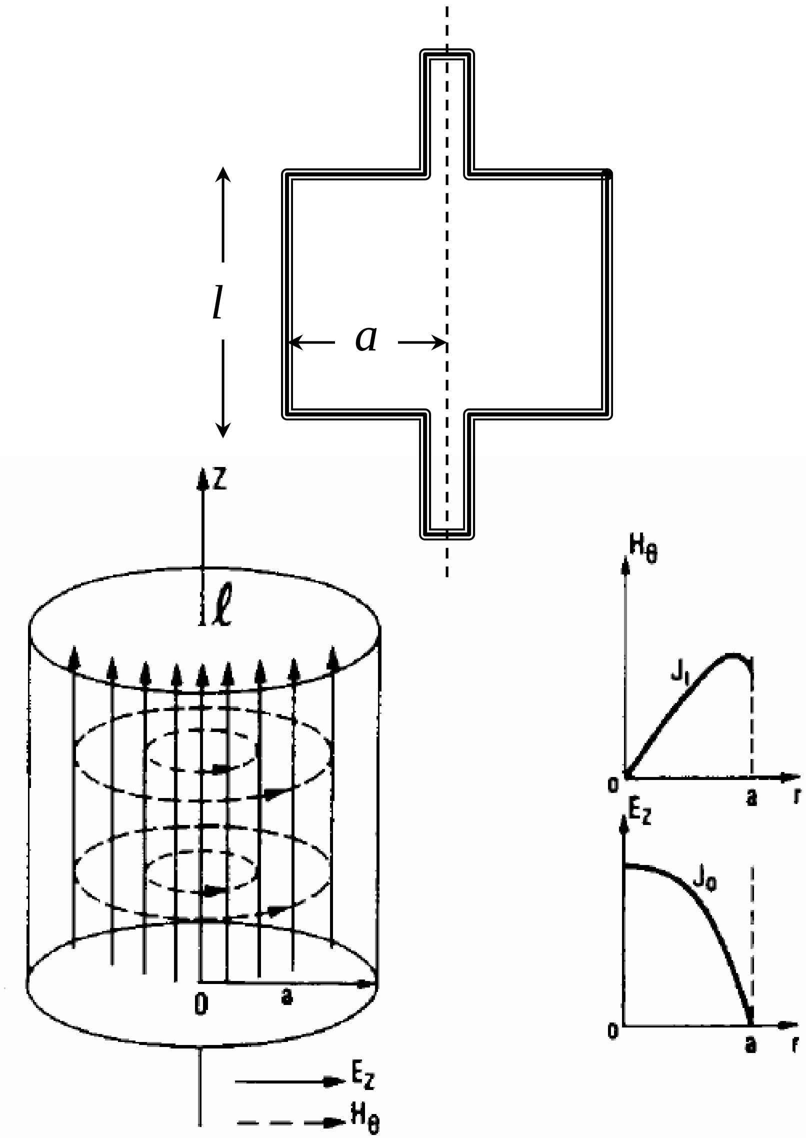
$$E_z = E_0 J_0(k_c r) e^{i(\omega t - kz)}$$

$$E_r = -iE_0 \frac{k_z}{k_c} J'_0(k_c r) e^{i(\omega t - kz)}$$

$$B_\theta = -i \frac{k}{k_c} E_0 J'_0(k_c r) e^{i(\omega t - kz)}$$

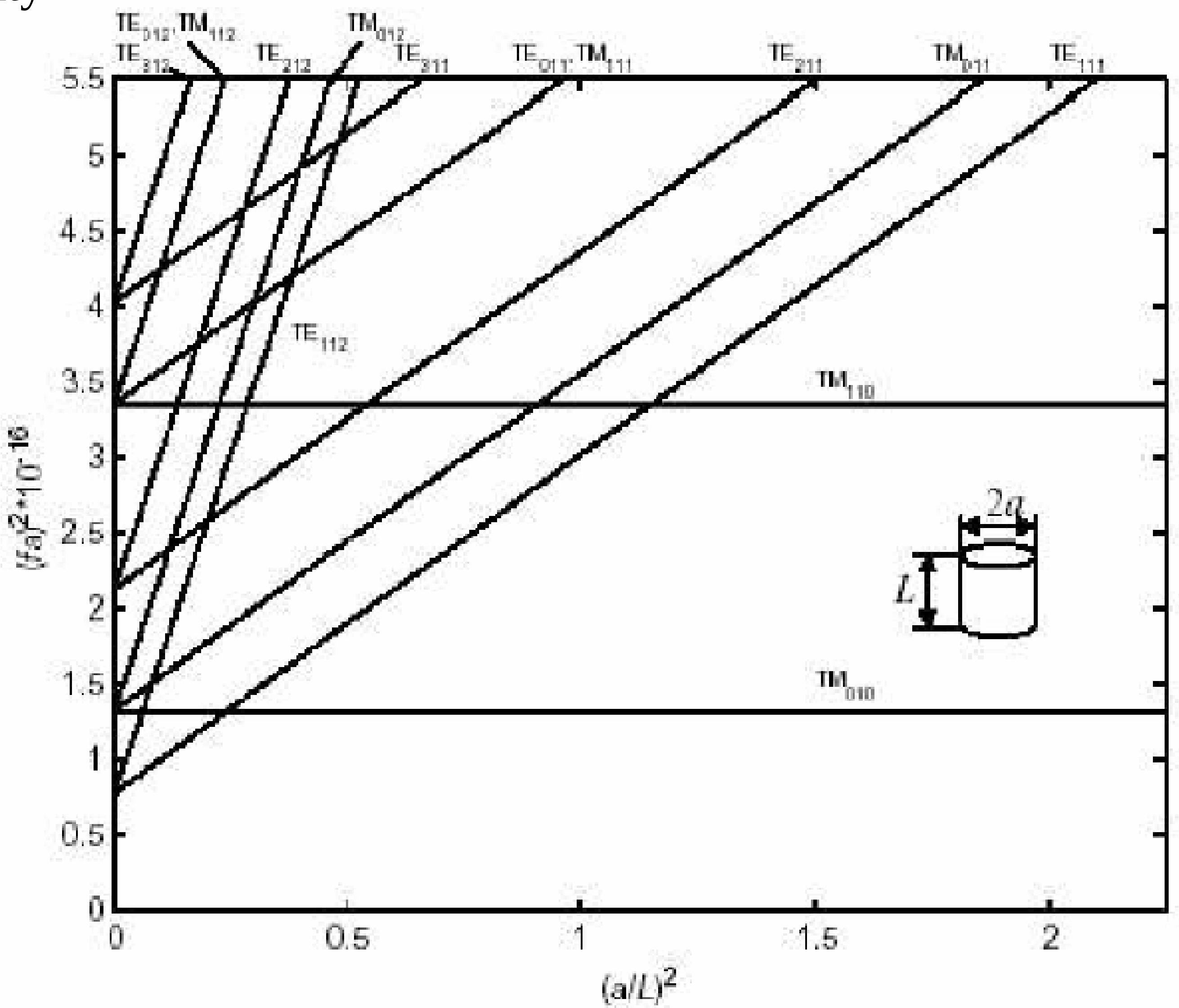
k_c and thus the frequency is fixed by the dimensions of the cavity

$$J_0(k_c a) = 0 \Rightarrow k_c a = 2.405$$



Mode frequency

Each mode has its resonant frequency defined by the geometry of the pillbox cavity



Cavity Parameters

Stored energy:

$$U = \frac{\varepsilon_0}{2}\int E^2 dV + \frac{\mu_0}{2}\int H^2 dV$$

The electric and magnetic stored energy oscillate in time 90 degrees out of phase. In practice, we can use either the electric or magnetic energy using the peak value.

Power dissipation:

$$P = \frac{R_s}{2}\int H^2 ds; \quad Rs = \frac{1}{\sigma\delta}; \quad \delta = \sqrt{\frac{2}{\sigma\mu_0\omega}}.$$

where R_s is the surface resistance, σ is the dc conductivity and δ is the skin depth

Quality factor:

$$Q = \omega\frac{U}{P} = \omega\frac{\text{cavity stored energy}}{\text{average power disspated}}$$

The pill box cavity II

Stored energy:

$$U = \frac{\pi}{2}\varepsilon_0 l a^2 E_0 J_1^{\,2}(2.405)$$

Power dissipation:

$$P = \pi \frac{\varepsilon_0}{\mu_0} a R_s E_0^{\,2} J_1^{\,2}(2.405)[l + a]$$

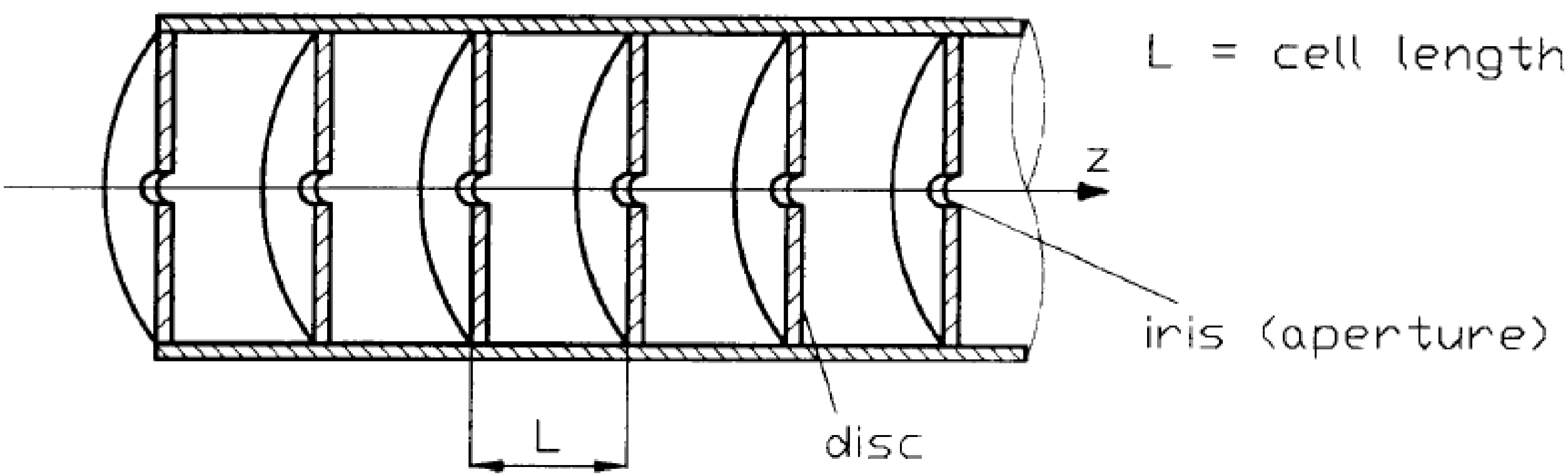
Quality factor:

$$Q = \omega \frac{U}{P} = \frac{\mu_0 c}{2R_s} \frac{2.405}{\left[1 + \frac{a}{l}\right]}$$

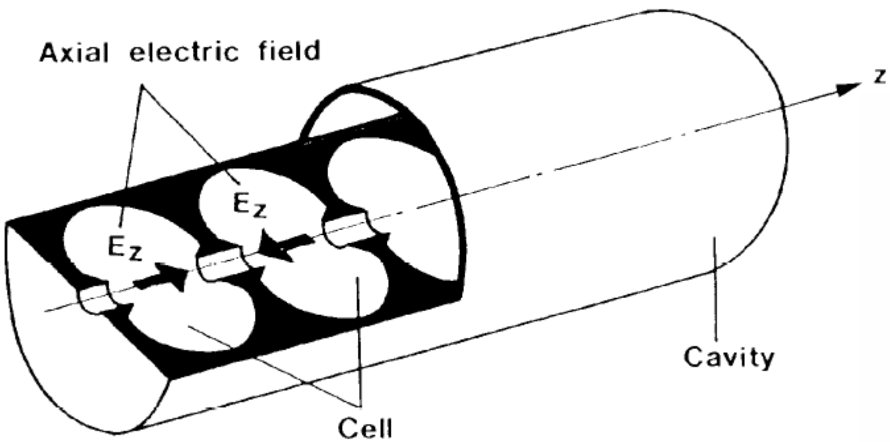
Loaded cavities

hints…

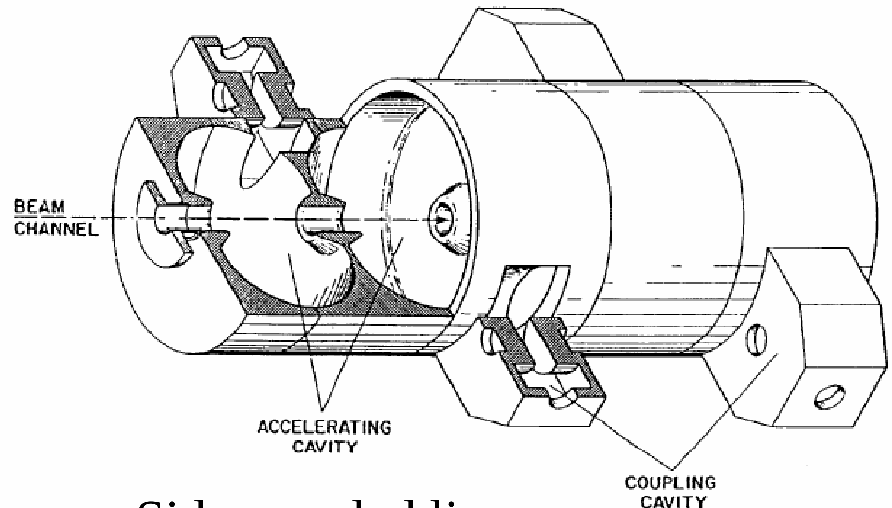
- We add a periodicity to the structure and change the boundary conditions
- Not a single mode but a whole spectrum is propagated thought the cavity
- The solution is now expressed in modified Bessel functions.
- We can have traveling wave accelerators and standing wave accelerators



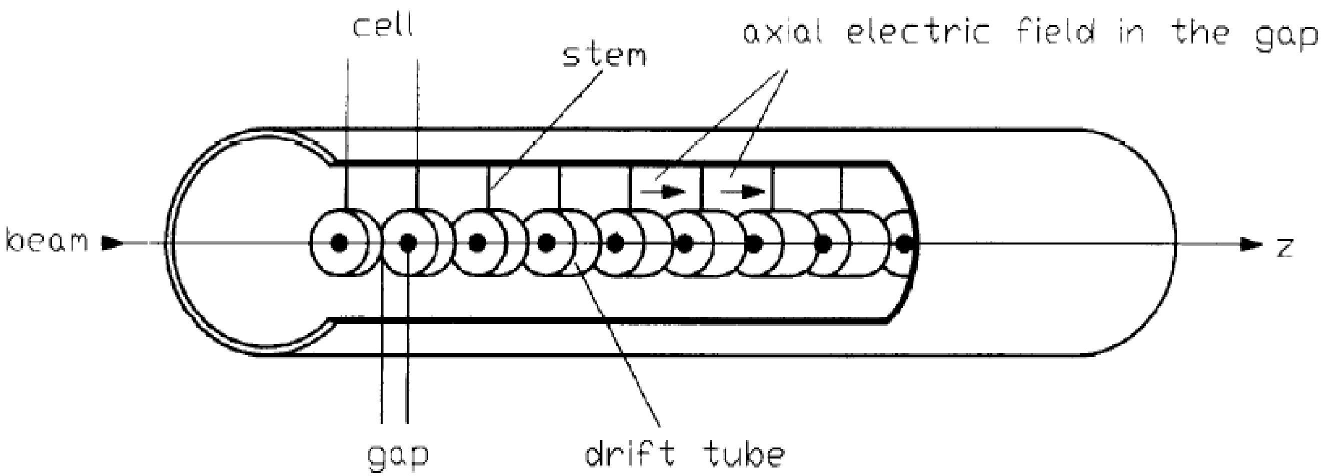
Other cavity types



Disc-loaded linac



Side-coupled linac



Drift tube linac (DTL)

The RFQ I

The RFQ uses only electric field to accelerate and focus the beam

The wave equation can be replaced with the Laplace equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial \theta^2}=0$$

The general solution

$$U(r,\theta,z)=\frac{V}{2}\left[\sum_n A_{0n}r^{2n}\cos 2n\theta+\sum_n\sum_l A_{ln}I_{2n}(lkr)\cos 2n\theta\cos lkz\right]$$

with l+n =2p+1 p=0,1,2…,V/2 the electrode potential, I_{2n} is the modified Bessel function of order 2n and k=2π/βλ

Taking only the low order solution

$$U(r,\theta,z)=\frac{V}{2}\left[A_{01}r^2\cos 2\theta+A_{10}I_0(kr)\cos kz\right]$$

The RFQ II

The first term is the potential of an electric quadrupole (focusing term);

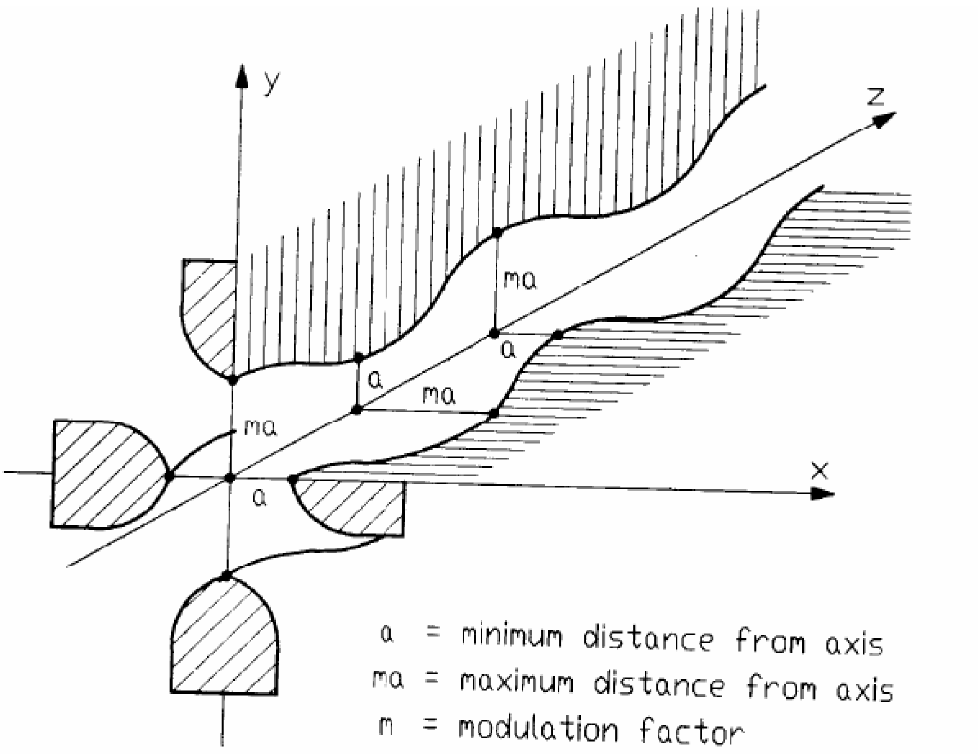
the second, will generate a longitudinal accelerating electric field.

Constants A_{01} and A_{10} are determined by imposing the voltage in the electrodes

$$A_{10} = \frac{m^2 - 1}{m^2 I_0(ka) + I_0(mka)},$$

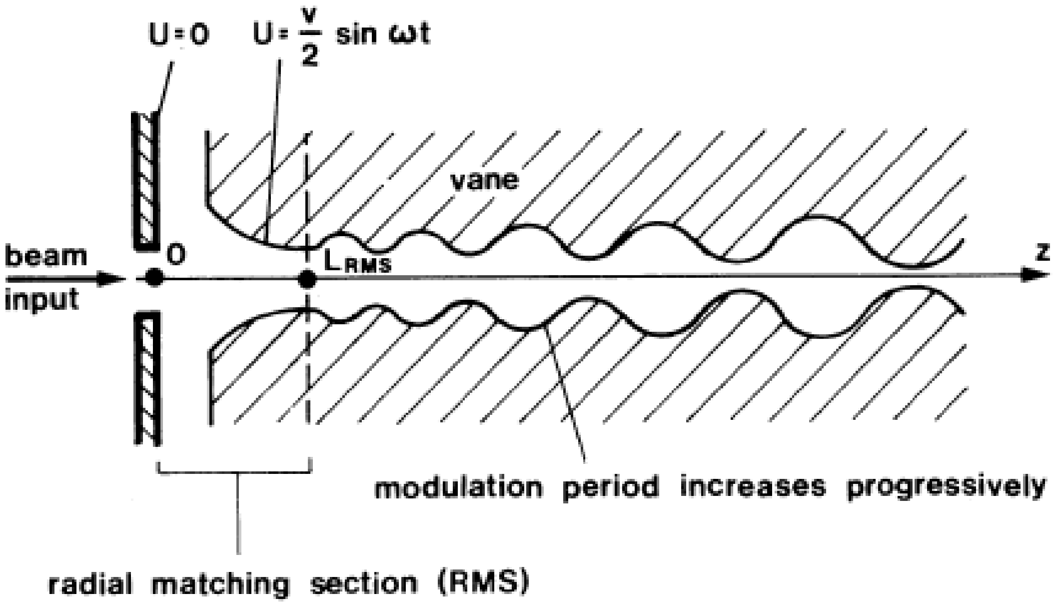
$$A_{01} = \frac{1}{a^2}\left[1 - A_{10} I_0(ka)\right] = \frac{\chi}{a^2}.$$

- Increasing *m* one gets more acceleration
- Decreasing *a* one gets more focusing



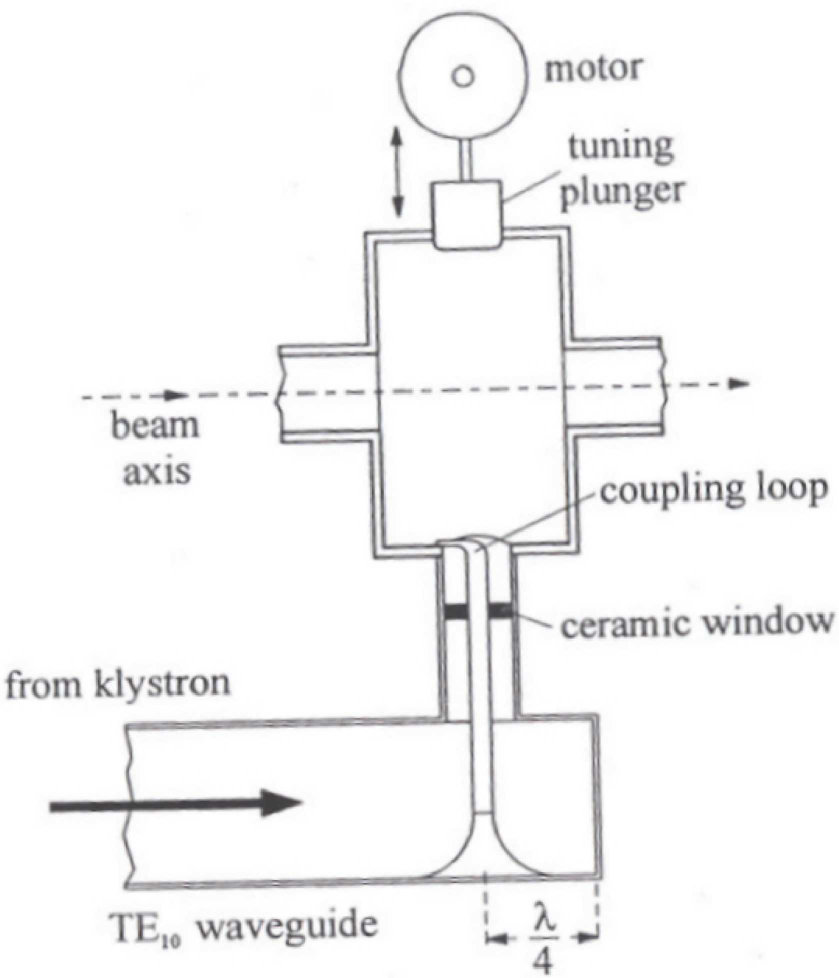
- The operation of the RFQ can best be understood by considering a long electric quadrupole with an alternating voltage on it,
- Particles moving along the z-axis and staying inside the RFQ for several periods of the alternating voltage, would be exposed to an alternating gradient focusing

- If the tips of the electrodes are not flat but 'modulated' a part of the electric field is 'deviated' into the longitudinal direction and this field can be used to bunch and accelerate particles



Feeding and tuning the cavity

- The transmission of the power between the generator and the cavity is done
  - through a coaxial line (short distances, low power < 100 kW)
  - through a waveguide. Low losses. Can be cooled
- The connection between the waveguide and the cavity is done with a short coaxial line with virtually no-losses
- A ceramic window inside the coaxial cable separates the waveguide from the cavity
- To bring the cavity into the resonance condition, tuning is done using tuning plungers



Bibliography

“*The physics of Particle Accelerators. An introduction*”, Klaus Wille Oxford university press ISBN 0 19 850549 3

“*Introduction to Linear Accelerators*”, Thomas P. Wangler. LA-UR-93-805

“Particle Accelerator Physiscs II”, H. Wiedemann Springer 1999

“*Dynamics and Acceleration in linear structures*”, J. le Duff and “*Conventional RF Systems Design*”, M. Puglisi in “*CERN Accelerator School: 5th General accelerator physics course*”, CERN 94-01

“*Fundamental of Ion Linacs*” M. Weiss, in “*CERN Accelerator School: Cyclotrons, linacs and their applications*”, CERN 96-02

Cut-off frequency

Dispersion or Brillouin diagram

As k_c is fixed by the geometry we are left with the equation of an hyperbola.

$$\left(\frac{\omega}{c}\right)^2 = k_c^2 + k_z^{\ 2}$$

The minimum frequency propagating in the cavity ωc is called the cut-off frequency