Phase space concepts

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Outline

- Transverse phase space
- Beam representation
- Beam Emittance
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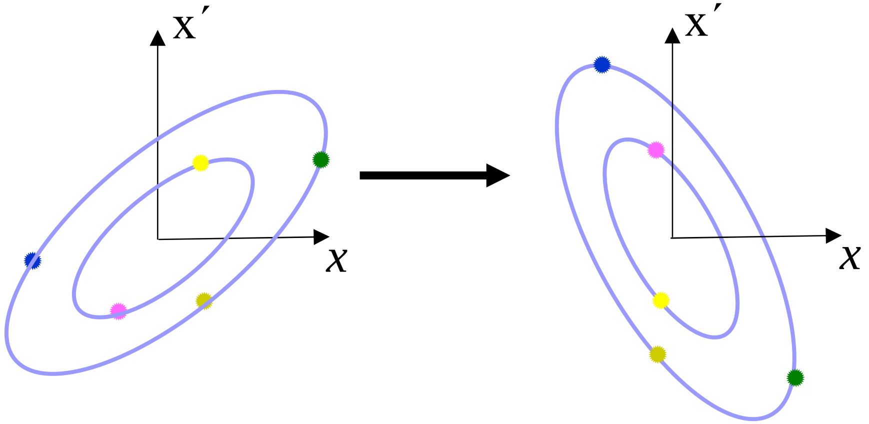
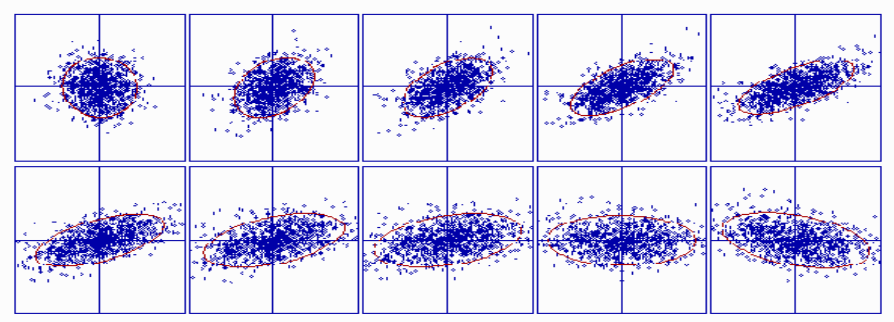
Transverse Phase Space

- Under linear forces, any particle moves on ellipse in phase space (x,x’), (y,y’).
- Ellipse rotates and moves between magnets, but its area is preserved.
- The area of the ellipse defines the **emittance**

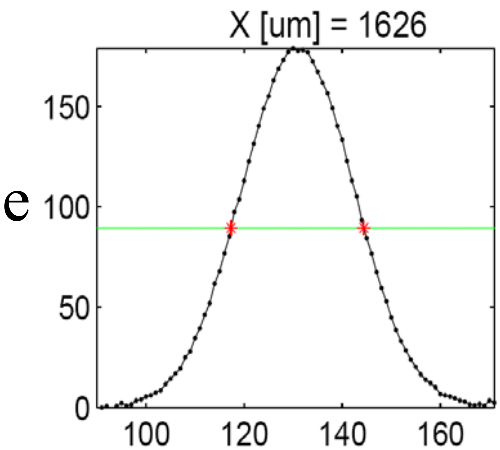


- The equation of the ellipse is
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$
with α,β,γ, the twiss parameters
- Due to large number of particles, need of a statistical description of the beam, and its size

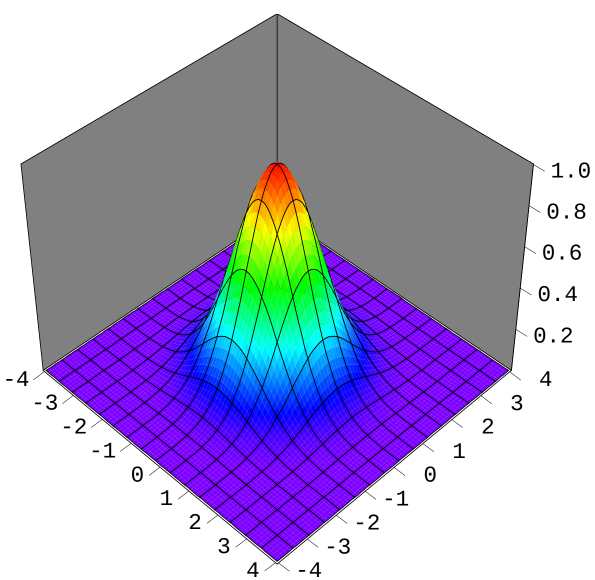


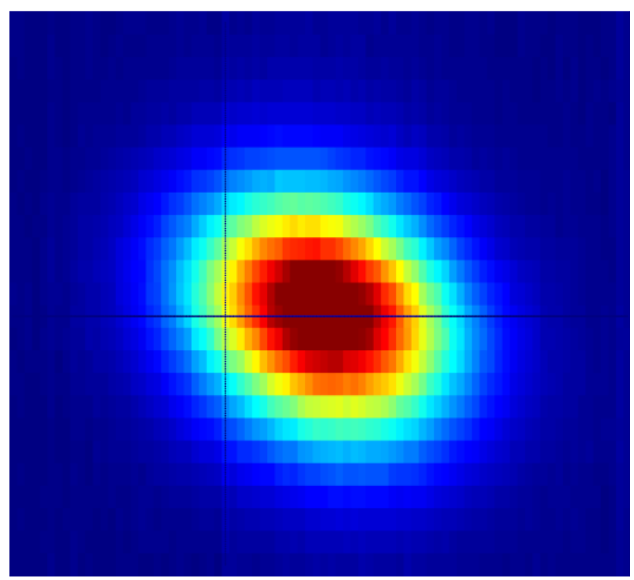
Beam representation

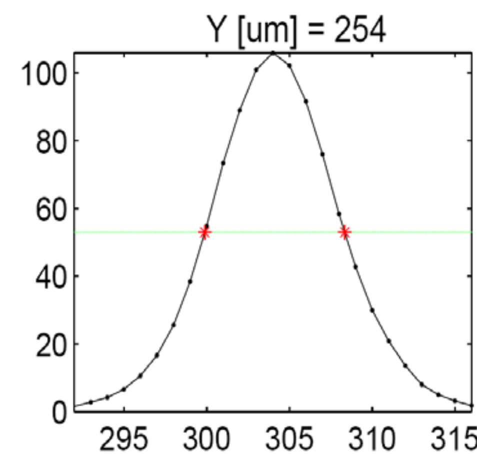
- Beam is a set of millions/billions of particles (N)
- A macro-particle representation models beam as a set of n particles with n<<N
- Distribution function is a statistical function representing the number of particles in phase space between $\mathbf{u} + d\mathbf{u}\ ,\quad \mathbf{u}' + d\mathbf{u}'$



$$f(\mathbf{u}, \mathbf{u}')d\mathbf{u}d\mathbf{u}' = \text{number of particles}$$







Liouville emittance

- Emittance represents the phase-space volume occupied by the beam
- The phase space can have different dimensions
  - 2D **(x, x')** or **(y, y')** or **(φ, E)**
  - 4D **(x, x',y, y')** or **(x, x', φ, E)** or **(y, y', φ, E)**
  - 6D **(x, x', y, y', φ, E)**
- The resolution of my beam observation is very large compared to the average distance between particles.
- The beam modeled by phase space **distribution function** $f(x, x', y, y', \phi, E)$
- The volume of this function on phase space is the beam **Liouville emittance**

Vlasov and Boltzmann equations

- The evolution of the distribution function is described by **Vlasov** equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = 0$$

- Mathematical representation of **Liouville theorem** stating the conservation of phase space volume $(\mathbf{q}, \mathbf{p})$
- In the presence of fluctuations (radiation, collisions, etc.) distribution function evolution described by **Boltzmann equation**

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{\gamma m_0}\frac{\partial f}{\partial \mathbf{q}} + \mathbf{F}(\mathbf{q})\frac{\partial f}{\partial \mathbf{p}} = \left.\frac{df}{dt}\right|_{\text{fluct}}$$

- The distribution evolves towards a **Maxwell-Boltzmann statistical equilibrium**

2D and normalized emittance

- When motion is uncoupled, Vlasov equation still holds for each plane individually

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p_u}{\gamma m_0}\frac{\partial f}{\partial u} + \mathbf{F}(u)\frac{\partial f}{\partial p} = 0$$

- The Liouville emittance in the 2D (u, p_u) phase space is still conserved
- In the case of acceleration, the emittance is conserved in the (u, p_u) but not in the (u, u') **(adiabatic damping)**
- Considering that

$$u' = \frac{du}{ds} = \frac{p_u}{p_s}$$

  the beam is conserved in the phase space $(u, u' p_s)$
- Define a **normalised emittance which is conserved during acceleration**

$$\epsilon_n = \beta_r \gamma_r \epsilon$$

Beam matrix

- We would like to determine the transformation of the beam enclosed by an ellipse through the accelerator
- Consider a vector **u = (x,x',y,y',…)** in a generalized n-dimensional phase space. In that case the ellipse transformation is

$$\mathbf{u}^T \cdot \mathbf{\Sigma}^{-1} \cdot \mathbf{u} = \mathcal{I}$$

- Application to one dimension gives $\Sigma_{11}u^2 + 2\Sigma_{22}uu' + \Sigma_{22}u'^2 = 1$ and comparing with $\gamma_u u^2 + 2\alpha_u uu' + \beta_u u'^2 = \epsilon_u$ provides the beam matrix $\Sigma_u = \begin{pmatrix} \beta_u & -\alpha_u \\ -\alpha_u & \gamma_u \end{pmatrix} \epsilon_u = \mathcal{B}\epsilon_u$ which can be expanded to more dimensions
- Evolution of the n-dimensional phase space from position 1 to position 2, through transport matrix $\mathcal{M}$

$$\mathcal{M} \cdot \mathbf{\Sigma}_1 \cdot \mathcal{M}^T = \mathbf{\Sigma}_2$$

Root Mean Square (RMS) beam parameters

- The average of a function on the beam distribution defined

$$\langle g(\mathbf{u},\mathbf{u}')\rangle = \frac{1}{n}\sum_{i=1}^{n} g(u_i,u_i') = \frac{1}{N}\iint f(\mathbf{u},\mathbf{u}')g(\mathbf{u},\mathbf{u}')d\mathbf{u}d\mathbf{u}'$$

- Taking the square root, the following **Root Mean Square (RMS)** quantities are defined
  - **RMS beam size**

$$u_{\rm rms} = \sqrt{\sigma_u} = \sqrt{\langle (u-\langle u\rangle)^2\rangle}$$

  - **RMS beam divergence**

$$u'_{\rm rms} = \sqrt{\sigma'_u} = \sqrt{\langle (u'-\langle u'\rangle)^2\rangle}$$

  - **RMS coupling**

$$(uu')_{\rm rms} = \sqrt{\sigma_{uu'}} = \sqrt{\langle (u-\langle u\rangle)(u'-\langle u'\rangle)\rangle}$$

RMS emittance

- Beam modelled as macro-particles
- Involved in processed linked to the statistical size
- The **rms emittance** is defined as

$$\epsilon_{\rm rms} = \sqrt{\langle u \rangle^2 \langle u' \rangle^2 - \langle u u' \rangle^2}$$

- It is a statistical quantity giving information about the minimum beam size
- For linear forces the rms emittance is conserved in the case of linear forces
- The determinant of the rms beam matrix $det(\Sigma_{\rm rms}) = \epsilon_{\rm rms}$
- Including acceleration, the determinant of 6D transport matrices is not equal to 1 but

$$\det(\mathcal{M}_{1\to 2}) = \sqrt{\frac{\beta_{r2}\gamma_{r2}}{\beta_{r1}\gamma_{r1}}}$$

Beam Twiss functions

- The best ellipse fitting the beam distribution is

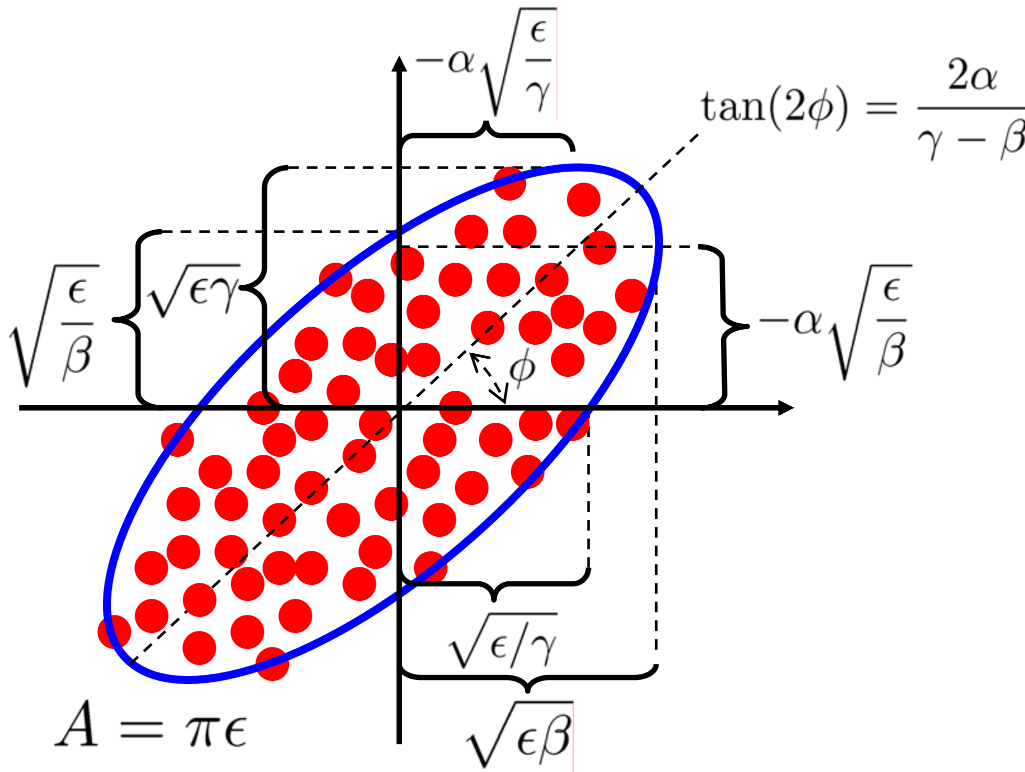
$$\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = \epsilon_u$$

- The beam Twiss functions can be defined through the rms emittance

$$\beta_u = \frac{u^2_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma_u}{\epsilon_{\rm rms}}$$

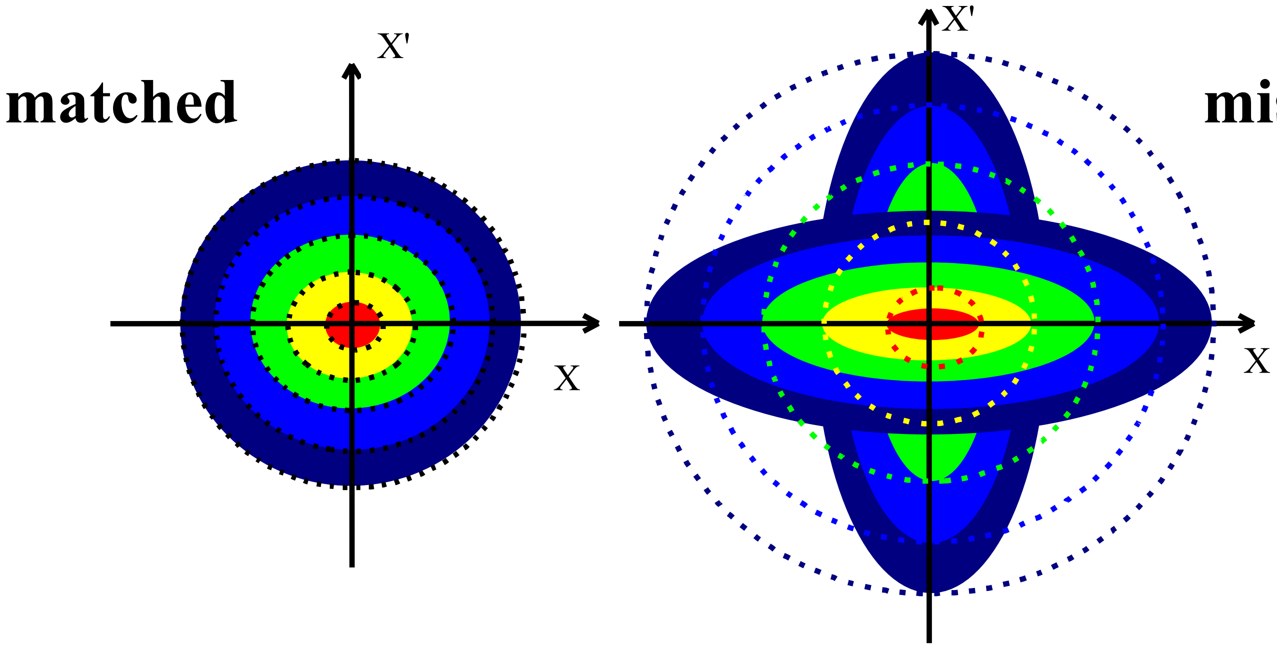
$$\gamma_u = \frac{u'^2_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma'_u}{\epsilon_{\rm rms}}$$

$$\alpha_u = \frac{(u\ u')_{\rm rms}}{\epsilon_{\rm rms}} = \frac{\sigma_{uu'}}{\epsilon_{\rm rms}}$$

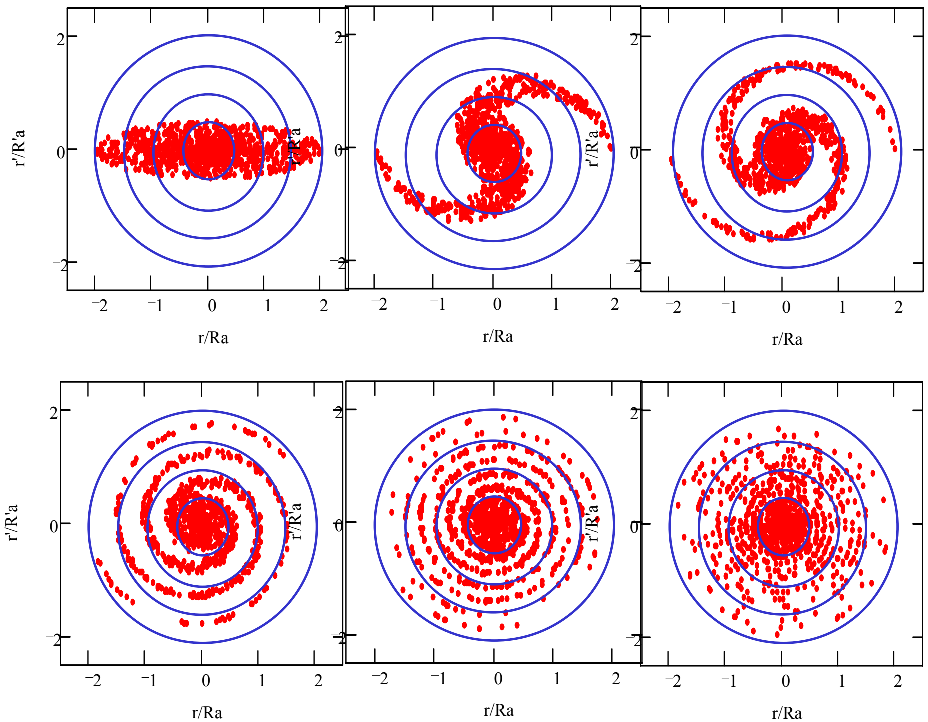


Matched beam

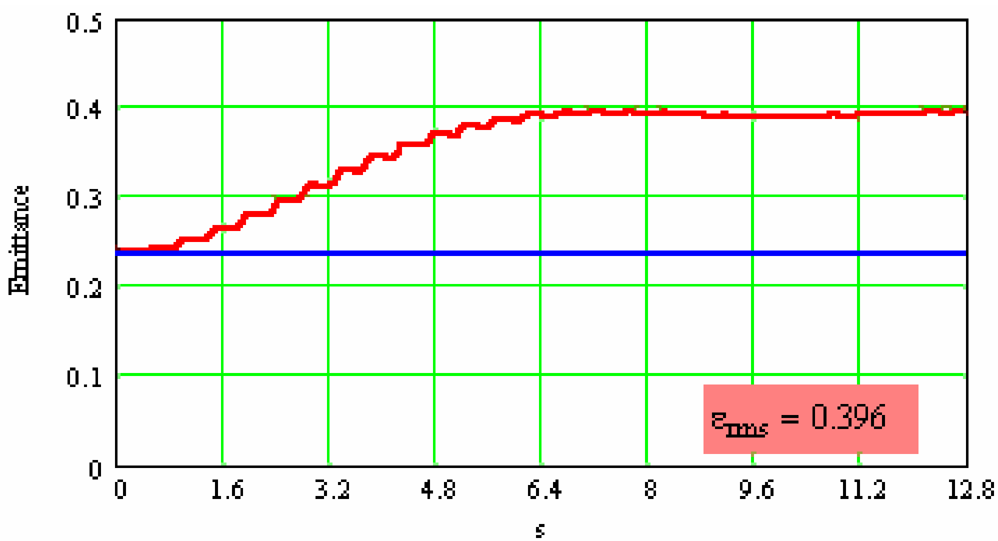
- The accelerator itself define ideal betatron functions for motion of individual particles on ellipses
- In order for the beam to be **matched** to the accelerator, the beam density should be constant on these ellipse
- In other words, the beam Twiss functions should be equal to the betatron functions imposed by the machine structure

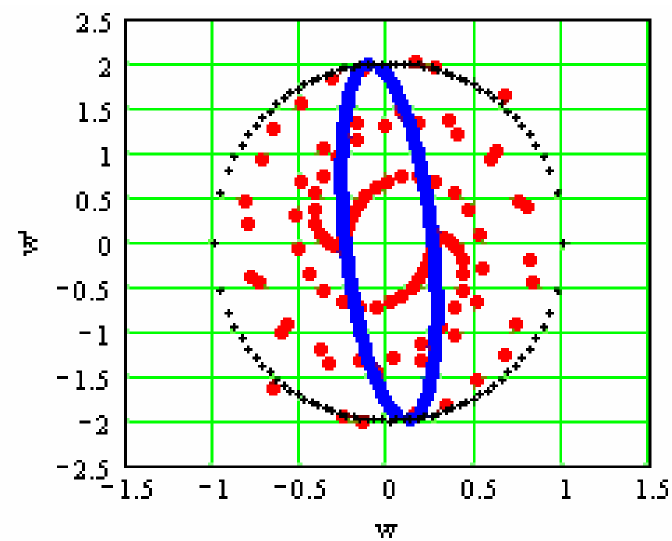


RMS emittance growth through filamentation



- When non-linear fields are present, particles in different positions in phase space have different frequency
- When the beam is rms mismatched, this gives **filamentation** leading to rms emittance growth





Linear beam distributions

Kapchinsky-Vladimirsky (KV) distribution

- K-V is a continuous beam whose distribution projection is uniform in 2D sub phase-spaces

$$f(x,x',y,y') = N\delta\left(\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{4\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{4\epsilon_{y,\mathrm{rms}}} - 2\right)$$

for which $\delta(u) = 0,\ \text{if } u \neq 0$ $\quad \int \delta\, dx dx' dy dy' = 1$

- The beam boundary is

$$\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = 4\epsilon_{u,\mathrm{rms}}$$

Waterbag distribution

- Spatial density decreases with radius but distribution is uniform in 4D phase space

$$f(x,x',y,y') = N\Gamma\left(\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{4\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{4\epsilon_{y,\mathrm{rms}}}\right)$$

- The beam boundary is $\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = 6\epsilon_{u,\mathrm{rms}}$

Gaussian distribution

- The **Gaussian distribution** has a gaussian density profile in phase space

$$f(x,x',y,y') = \frac{N}{A} exp\left(-\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{2\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{2\epsilon_{y,\mathrm{rms}}}\right)$$

for which $\int f(\mathbf{u},\mathbf{u}') d\mathbf{u} d\mathbf{u}' = N$

  - The beam boundary is $\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = n^2 \epsilon_{u,\mathrm{rms}}$

KV | **4D Waterbag** | **6D Waterbag** | **Gaussian**