Principles of charged particle beam optics

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Outline

- Particle motion in circular accelerator
  - Coordinate system
  - Beam guidance
    - Dipoles
  - Beam focusing
    - Quadrupoles
  - Equations of motion
  - Multipole field expansion

Equation reminder

- Lorentz equation

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- E_{tot}: Total energy
  T: Kinetic energy

$$E_{tot}^2 = p^2c^2 + m_0^2c^4 = (T + m_0c^2)^2$$

- β: reduced velocity
  γ: reduced energy
  βγ: reduced momentum

$$\beta = \frac{v}{c} \qquad \gamma = \frac{E}{m_0c^2}$$

$$\beta\gamma = \frac{p}{mc}$$

Electric and magnetic field

- Only electric field accelerates particles

$$\frac{dE}{dt} = \mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\mathbf{v} \cdot \mathbf{E}$$

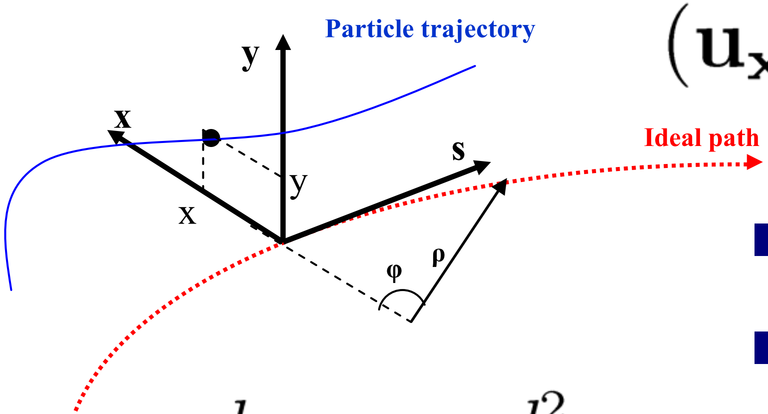
- Lorentz equation for **x** deviation of particle moving along **z** direction

$$\frac{dp_x}{dt} = \mathbf{F_x} = q(E_x - v_z B_y)$$

- In order to have no acceleration $E_x = v_z B_y$
- For relativistic particles $E_x >> B_y$
- Magnetic field is used for guiding particles (except special cases in very low energies)

Coordinate system

- Cartesian coordinates not useful to describe motion in an accelerator
- Instead we use a system following an ideal path along the accelerator



$$(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z}) \rightarrow (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_s})$$

- The curvature is $\mathbf{k} = -\frac{d^2\mathbf{s}}{ds^2}$
- From Lorentz equation

$$\frac{d\mathbf{p}}{dt} = m\gamma\frac{d^2\mathbf{s}}{dt^2} = m\gamma v_s^2\frac{d^2\mathbf{s}}{ds^2} = -m\gamma v_s^2\mathbf{k} = q|\mathbf{v}\times\mathbf{B}|$$

- The ideal path is defined $\mathbf{k} = -\frac{q}{p}|\frac{\mathbf{v}}{v_s}\times\mathbf{B}|$

Beam guidance

- Consider only a uniform magnetic field **B** in the direction perpendicular to the particle motion. From the ideal trajectory and after considering that the transverse velocities $\mathbf{v_x} \ll \mathbf{v_s}, \mathbf{v_y} \ll \mathbf{v_s}$, we have that the radius of curvature is

$$\frac{1}{\rho} = |k| = |\frac{q}{p}B| = |\frac{q}{\beta E_{tot}}B|$$

- The cyclotron or **Larmor frequency**

$$\omega_L = |\frac{qc}{E_{tot}}B|$$

- We define the **magnetic rigidity**

$$|B\rho| = \frac{p}{q}$$

- In more practical units

$$\beta E_{tot}[GeV] = 0.2998|B\rho|[Tm]$$

- For ions with charge multiplicity **Z** and atomic number **A**, the energy per nucleon is

$$\beta \bar{E}_{tot}[GeV/u] = 0.2998\frac{Z}{A}|B\rho|[Tm]$$

Dipoles

- Consider a storage ring for particles with energy **E** with **N** dipoles of length **l**
- The bending angle is

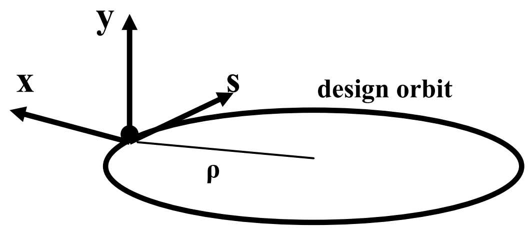
$$\theta = \frac{2\pi}{N}$$

- The bending radius is

$$\rho = \frac{l}{\theta}$$

- The integrated dipole strength will be $Bl = \frac{2\pi}{N}\frac{\beta E}{q}$
- By fixing the dipole field, the dipole length is imposed and vice versa
- The highest the field, shorter or smaller number of dipoles can be used
- Ring circumference (cost) is influenced by the field choice

Beam focusing



- Consider a particle in the design orbit.
- In the horizontal plane it does an harmonic oscillation

$$x = x_0 \cos(\omega t + \phi) \quad \text{with frequency} \quad \omega = \frac{v_s}{\rho}$$

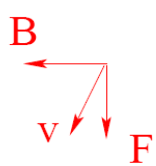
- The horizontal acceleration is described by

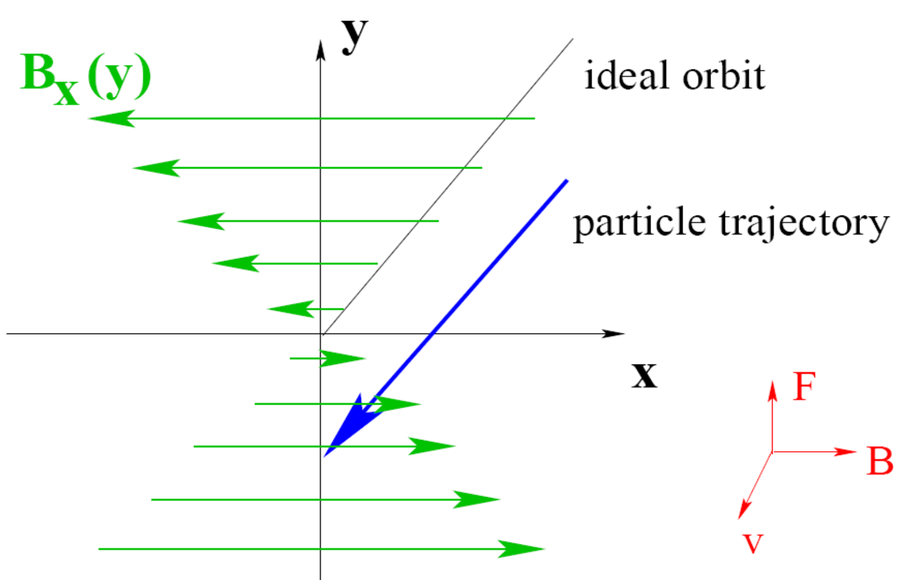
$$\frac{d^2x}{ds^2} = \frac{d^2x}{v_s^2 dt^2} = -\frac{1}{\rho^2}x$$

- There is a **week focusing** effect in the horizontal plane.
- In the vertical plane, only force is gravitation. The particle will be displaced vertically following the usual law

$$\Delta y = \frac{1}{2} g \Delta t^2$$

- Setting g = 10 m/s^2, the particle will be displaced by **18mm** (LHC dipole aperture) in **60ms** (a few hundreds of turns in LHC)
- Need of **focusing**!





Focusing elements

- Magnetic element that deflects the beam by an angle proportional to the distance from its centre (equivalent to ray **optics**) provides focusing.
- For a focal length f the deflection angle is $\alpha = -\frac{y}{f}$
- A magnetic element with length **l** and with a gradient **g** has a field $B_x = gy$ so that the deflection angle is

$$\alpha = -\frac{l}{f} = -\frac{q}{\beta E} B_x l = -\frac{q}{\beta E} g y l$$

- The normalised focusing strength

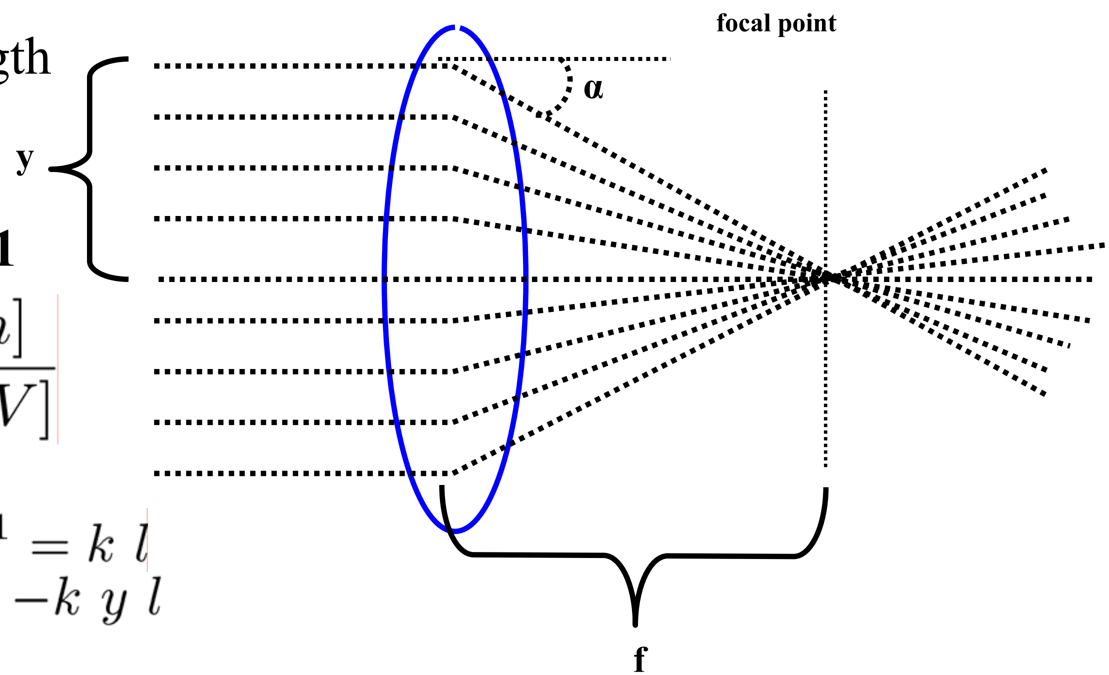
$$k = \frac{qg}{\beta E}$$

- In more practical units, for **Z=1**

$$k[m^{-2}] = 0.2998 \frac{g[T/m]}{\beta E[GeV]}$$

- The focal length becomes $f^{-1} = k\ l$

and the deflection angle is $\alpha = -k\ y\ l$

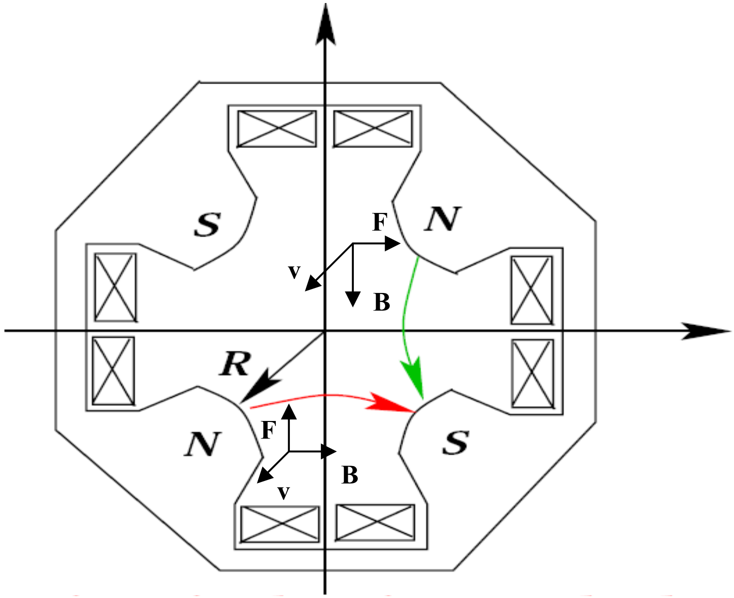


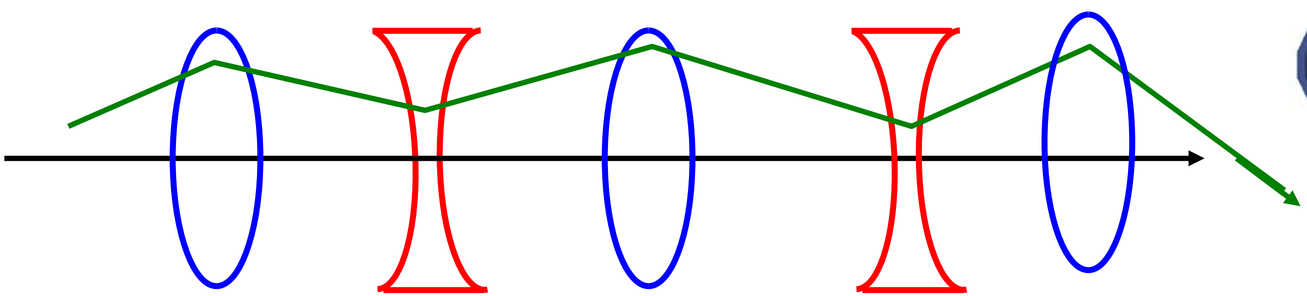
Quadrupoles

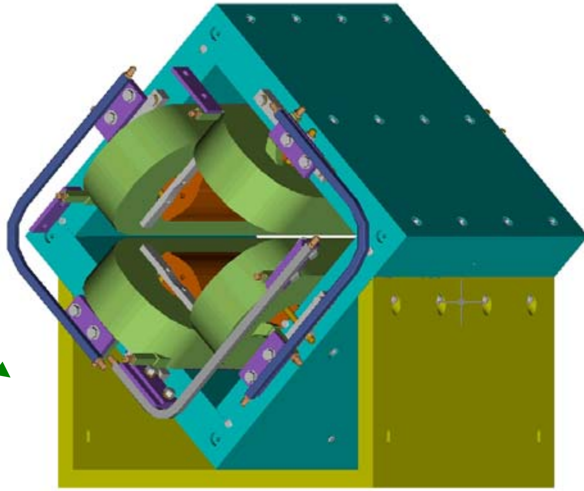
- Quadrupoles are focusing in one plane and defocusing in the other
- The field is $(B_x, B_y) = g(y, x)$
- The resulting force $(F_x, F_y) = k(y, -x)$
- Need to alternate focusing and defocusing in order to control the beam, i.e. **alternating gradient focusing**
- From optics we know that a combination of two lenses with focal lengths **f1** and **f2** separated by a distance **d**

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\ f_2}$$

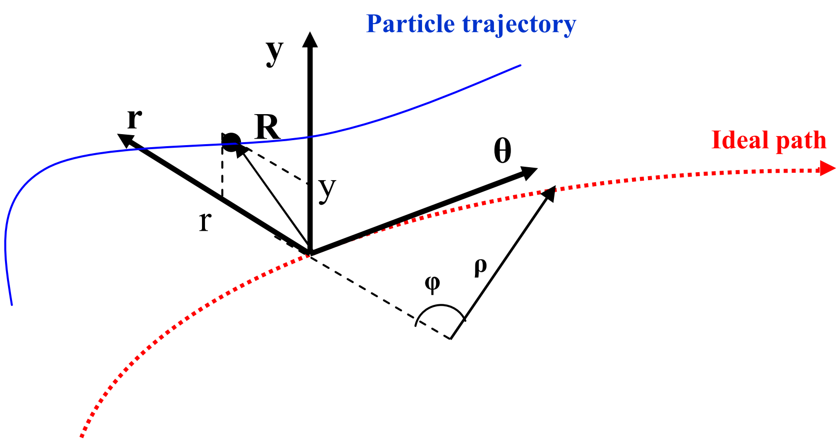
- If $f_1 = -f_2$, there is a net focusing effect, i.e. $\frac{1}{f} = |\frac{d}{f_1\ f_2}|$







Rotating coordinate system



- Consider a particle with charge **q** moving in the presence of transverse magnetic fields
- Choose cylindrical coordinate system **(r,φ,y)**, with **r = x+ρ** and **φ = s/ρ**
- The radius vector is

$$\mathbf{R} = \mathbf{R_0} + r\mathbf{u_r} + y\mathbf{u_y}$$

- For a small displacement **dφ**

$$d\mathbf{u_r} = d\phi\mathbf{u}_\phi \ , \quad d\mathbf{u}_\phi = -d\phi\mathbf{u_r} \ , \quad d\mathbf{u_y} = 0$$

- Than the velocity is $\dot{\mathbf{R}} = \dot{r}\mathbf{u_r} + r\dot{\phi}\mathbf{u}_\phi + \dot{y}\mathbf{u_y}$

- And the acceleration $\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\phi}^2)\mathbf{u_r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\mathbf{u}_\phi + \ddot{y}\mathbf{u_y}$

- Recall that the momentum is $\dot{\mathbf{p}} = \frac{d}{dt}(\gamma m_0\dot{\mathbf{R}}) = \gamma m_0\ddot{\mathbf{R}}$

Equations of motion

- Setting the electric field to zero and the magnetic field

$$\mathbf{B} = (B_r, B_\phi, B_y) = (B_x, 0, B_y)$$

The Lorentz equations become

$$\dot{\mathbf{p}} = q\mathbf{v} \times \mathbf{B} = r\dot{\phi}\mathbf{u_r} + (\dot{y}B_x - \dot{r}B_y)\mathbf{u}_\phi - r\dot{\phi}B_x\mathbf{u_y}$$

- Replacing the momentum with the adequate expression and splitting the equations for the **r** and **y** direction

$$\begin{aligned} \gamma m_0(\ddot{r} - r\dot{\phi}^2) &= -qr\dot{\phi}B_y \\ \gamma m_0 \ddot{y} &= qr\dot{\phi}B_x \end{aligned}$$

- Replace $r\dot{\phi} = v_\phi$, $r = x + \rho$ and as $v_\phi >> v_r,\ v_y \rightarrow P/v_\phi \approx \gamma m_0$
- The equations of motion in the new coordinates are

$$\begin{aligned} \frac{P}{v_\phi}(\ddot{x} - \frac{v_\phi^2}{\rho + x}) &= -qv_\phi B_y \\ \frac{P}{v_\phi}\ddot{y} &= qv_\phi B_x \end{aligned}$$

General equations of motion

- Note that for **x<<ρ** $$\frac{1}{\rho + x} = \frac{1}{\rho}(1 - \frac{x}{\rho})$$
- It is convenient to consider the arc length **s** as the independent variable

$$ds = \rho d\phi = \rho\dot{\phi}dt = v_\phi \frac{\rho}{\rho + x} dt \approx v_\phi (1 - \frac{x}{\rho}) dt$$

and $$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_\phi (1 - \frac{x}{\rho})\frac{d}{ds} \ , \quad \frac{d^2}{dt^2} \approx v_\phi^2 \frac{d^2}{ds^2}$$

- Denote $$\frac{dx}{ds} = x' \ , \quad \frac{d^2 x}{ds^2} = x''$$
- The general equations of motion are

$$\begin{aligned} x'' &= \frac{1}{\rho}(1 - \frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' &= \frac{qB_x}{P} \end{aligned}$$

- **Remark:** Note that without the approximations, the equations are nonlinear and coupled!
- The fields have to be defined

Magnetic multipole expansion

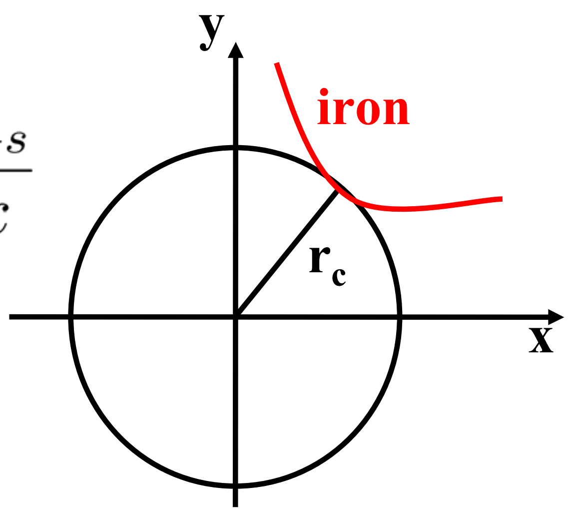
- From Gauss law of magnetostatics, we construct a vector potential

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming a 2D field in **x** and **y**, the vector potential has only one component $\mathbf{A_s}$
- The Ampere's law in vacuum (inside the beam pipe)

$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations one finds the conditions which are Riemann conditions of an analytic function.



$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

- There exist a complex potential of **z = x+iy** with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Magnetic multipole expansion

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \; , \quad a_n = n\mu_n$ we have

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized units $b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \; , \; a_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$

on a reference radius, 10^{-4} of the main field to get

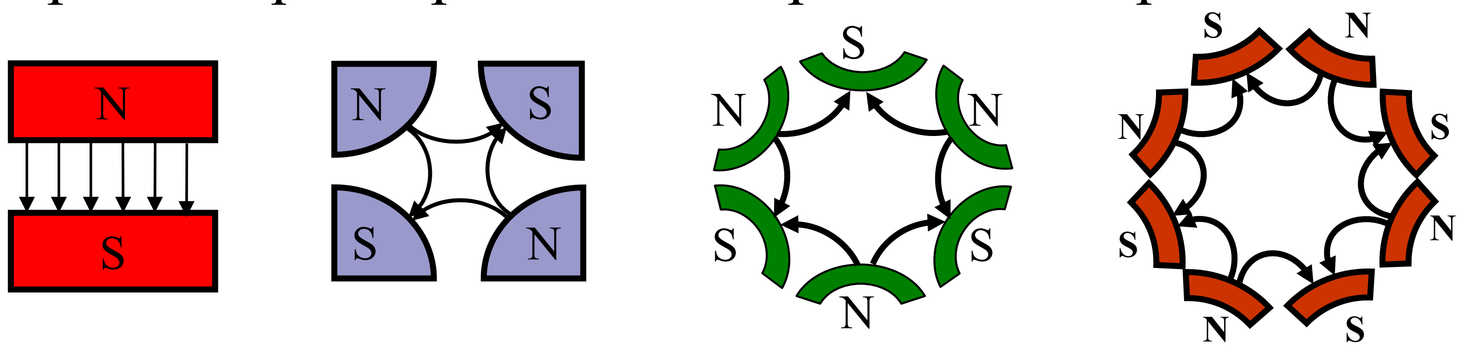
$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note**: **n'=n-1** the American convention

Magnet definitions

- **2n-pole**:

dipole quadrupole sextupole octupole …



n: 1 2 3 4 …

- Normal: gap appears at the horizontal plane
- Skew: rotate around beam axis by **π/2n** angle
- Symmetry: rotating around beam axis by **π/n** angle, the field is reversed (polarity flipped)