Principles of charged particle beam optics

Yannis PAPAPHILIPPOU

JUAS, Archamps, FRANCE

11-12 January 2007

Outline

- Particle motion in circular accelerators
  - Notions of charged particle dynamics and coordinate systems
  - Guidance and focusing
  - Equations of motion
  - Multipole field expansion
- Accelerator magnets
  - Magnetic materials
  - Magnetic potential and symmetries
- Hill's equations
  - Equations of motions in linear magnetic fields
- Transport Matrices
  - Matrix formalism
  - Drift
  - Thin lens approximation
  - Quadrupoles
  - Dipoles
    - Sector magnets
    - Rectangular magnets
  - Doublet, FODO cell
- Off-momentum particles
  - Effect from dipoles and quadrupoles
  - Dispersion equation
  - 3x3 transfer matrices
  - Momentum compaction and transition energy

References

- O. Bruning, Accelerators, Summer student courses, 2005.
- J. Rossbach and P. Schmuser, Basic course on accelerator optics, CERN Accelerator School, 1992.
- H. Wiedemann, Particle Accelerator Physics I, Springer, 1999.
- K.Wille, The physics of Particle Accelerators, Oxford University Press, 2000.

Equation reminder

Lorentz equation with
p, **v**: relativistic momentum and velocity vectors
F: electromagnetic force vector
E, **B**: electric and magnetic field vectors
q: electric charge

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

E_{tot}: total energy
T: kinetic energy
m_0: rest mass
c: speed of light

$$E_{tot}^2 = p^2c^2 + m_0^2c^4 = (T + m_0c^2)^2$$

β, γ, $\beta\gamma$: reduced velocity, energy and momentum

$$\beta = \frac{v}{c} \qquad \gamma = \frac{E_{tot}}{m_0c^2} \qquad \beta\gamma = \frac{p}{m_0c}$$

Notions of particle beam dynamics

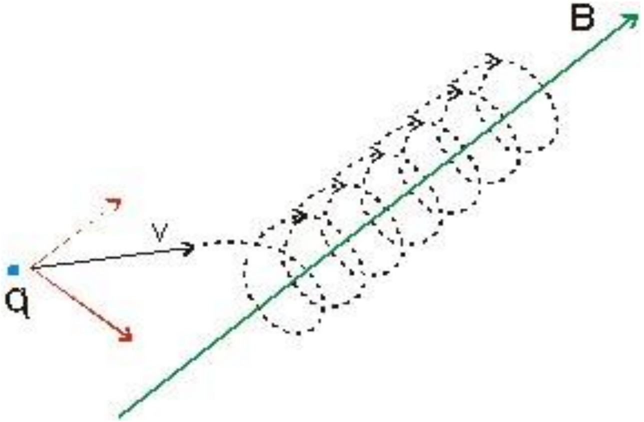
Starting from the definition of the relativistic momentum $\mathbf{p} = m_0\gamma\mathbf{v}$
the particle acceleration is $\dot{\mathbf{p}} = (m_0\gamma\mathbf{v}) = m_0\gamma\dot{\mathbf{v}} + m_0\mathbf{v}\dot{\gamma}$ with
$\dot{\gamma} = \gamma^3\dot{v}\beta/c$, giving $\dot{\mathbf{p}} = m_0(\gamma\dot{\mathbf{v}} + \gamma^3\beta\mathbf{v}\dot{v}/c) = \dot{\mathbf{p}}_{||} + \dot{\mathbf{p}}_{\perp}$

$$\dot{\mathbf{p}}_{\perp} = m_0\gamma\dot{\mathbf{v}}_{\perp} \qquad \dot{\mathbf{p}}_{||} = m_0\gamma^3\dot{\mathbf{v}}_{||}$$

Force perpendicular to propagation — Force parallel to propagation

In conclusion, **parallel acceleration** is much more efficient

- Example: Motion in a uniform, constant magnetic field
  - Spiraling along the magnetic field with constant energy. Equate centrifugal force with Lorentz force to get $\frac{m_0\gamma v^2}{\rho} = q\,v\,B$



from where the radius is $\rho = \frac{m_0\gamma v}{qB} = |\frac{p}{qB}|$

and the frequency $\omega = \frac{\nu}{\rho} = \frac{qB}{m_0\gamma} = \frac{qBc^2}{E}$

Electric and magnetic field

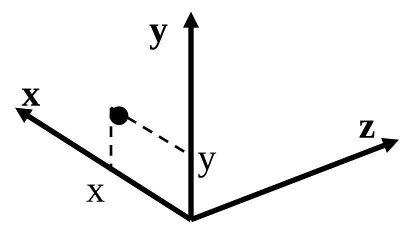
- Using Lorentz equation, the change in kinetic energy (work done by the Lorentz force over the path **s**) is

$$\Delta T = \int \mathbf{F} d\mathbf{s} = q \int \mathbf{E} d\mathbf{s} + q \cancel{\int (\mathbf{v} \times \mathbf{B}) \mathbf{v} dt}$$

i.e. **electric field is used for accelerating particles**

- Lorentz equation for **x** deviation of particle moving along **z** direction

$$\frac{dp_x}{dt} = \mathbf{F_x} = q(E_x - v_z B_y)$$



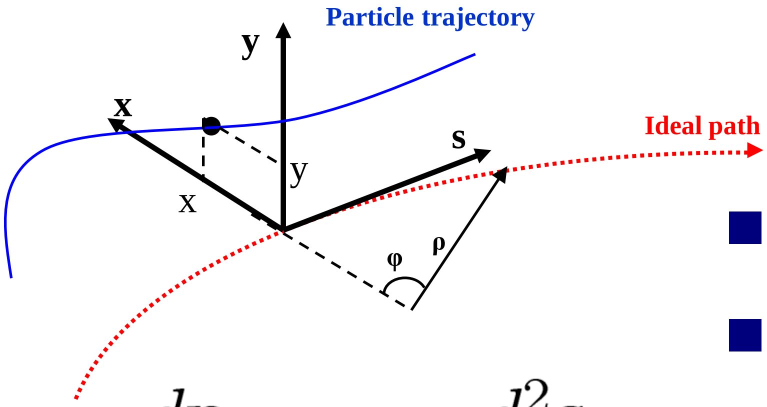
- For no acceleration $E_x - v_z B_y = 0 \Leftrightarrow E_x = v_z B_y$
- For relativistic particles, $v_z \approx c \Rightarrow E_x >> B_y$

i.e. **magnetic field is used for guiding particles** (except special cases in very low energies)

Coordinate system

- Cartesian coordinates not useful to describe motion in an accelerator

- Instead, a moving coordinate system following an ideal path along the accelerator is used (**Frenet** reference frame)



$$(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z}) \longrightarrow (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_s})$$

- The curvature vector is $\mathbf{k} = -\dfrac{d^2\mathbf{s}}{ds^2}$
- From Lorentz equation

$$\frac{d\mathbf{p}}{dt} = m_0\gamma\frac{d^2\mathbf{s}}{dt^2} = m_0\gamma v_s^2\frac{d^2\mathbf{s}}{ds^2} = -m_0\gamma v_s^2\mathbf{k} = q|\mathbf{v}\times\mathbf{B}|$$

- The ideal path is defined $\mathbf{k} = -\dfrac{q}{p}|\dfrac{\mathbf{v}}{v_s}\times\mathbf{B}|$

Beam guidance

- Consider only a uniform magnetic field ***B*** in the direction perpendicular to the particle motion. From the ideal trajectory and after considering that the transverse velocities $v_x \ll v_s, v_y \ll v_s$, we have that the radius of curvature is

$$\frac{1}{\rho} = |k| = |\frac{q}{p}B| = |\frac{q}{\beta E_{tot}}B|$$

- The **cyclotron** or **Larmor frequency** $$\omega_L = |\frac{qc^2}{E_{tot}}B|$$

- We define the **magnetic rigidity** $$|B\rho| = \frac{p}{q}$$

- In more practical units $$\boxed{\beta E_{tot}[GeV] = 0.2998|B\rho|[Tm]}$$

- For ions with charge multiplicity ***Z*** and atomic number ***A***, the energy per nucleon is

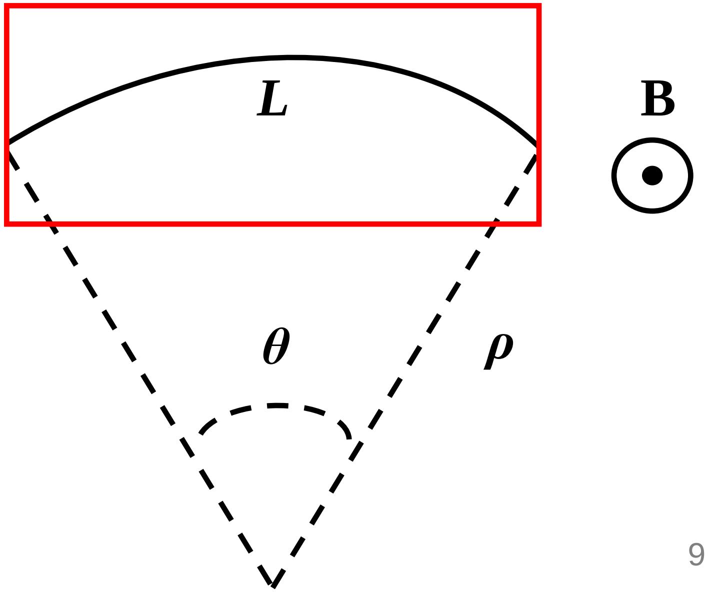
$$\beta \bar{E}_{tot}[GeV/u] = 0.2998\frac{Z}{A}|B\rho|[Tm]$$

Dipoles

- Consider an accelerator ring for particles with energy ***E*** with ***N*** dipoles of length ***L***

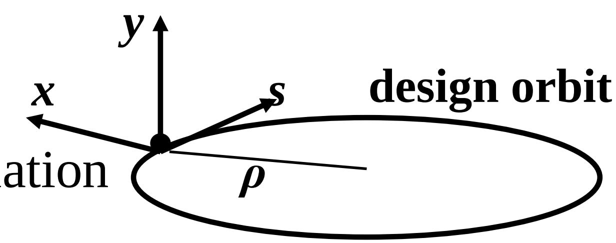
- **Bending angle** $\theta = \frac{2\pi}{N}$
- **Bending radius** $\rho = \frac{L}{\theta}$
- **Integrated dipole strength**

$$BL = \frac{2\pi}{N}\frac{\beta E}{q}$$



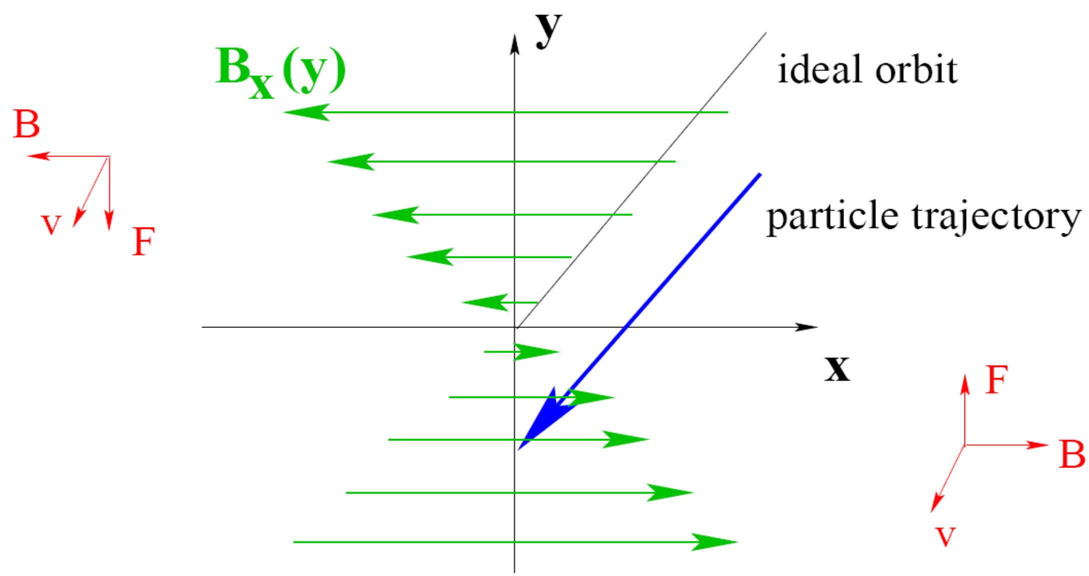
- Comments:
  - By choosing a dipole field, the dipole length is imposed and vice versa
  - The higher the field, shorter or smaller number of dipoles can be used
  - Ring circumference (cost) is influenced by the field choice

Beam focusing



- Consider a particle in the design orbit.
- In the horizontal plane it performs an harmonic oscillation $x = x_0 \cos(\omega t + \phi)$ with frequency $\omega = \frac{v_s}{\rho}$
- The horizontal acceleration is described by $\frac{d^2x}{ds^2} = \frac{1}{v_s^2}\frac{d^2x}{dt^2} = -\frac{1}{\rho^2}x$
- There is a **week focusing** effect in the horizontal plane.
- In the vertical plane, only force is gravitation. The particle will be displaced vertically following the usual law $\Delta y = \frac{1}{2} g \Delta t^2$
- Setting g = 10 m/s^2, the particle will be displaced by **18mm** (LHC dipole aperture) in **60ms** (a few hundreds of turns in LHC)

Need of **strong focusing**



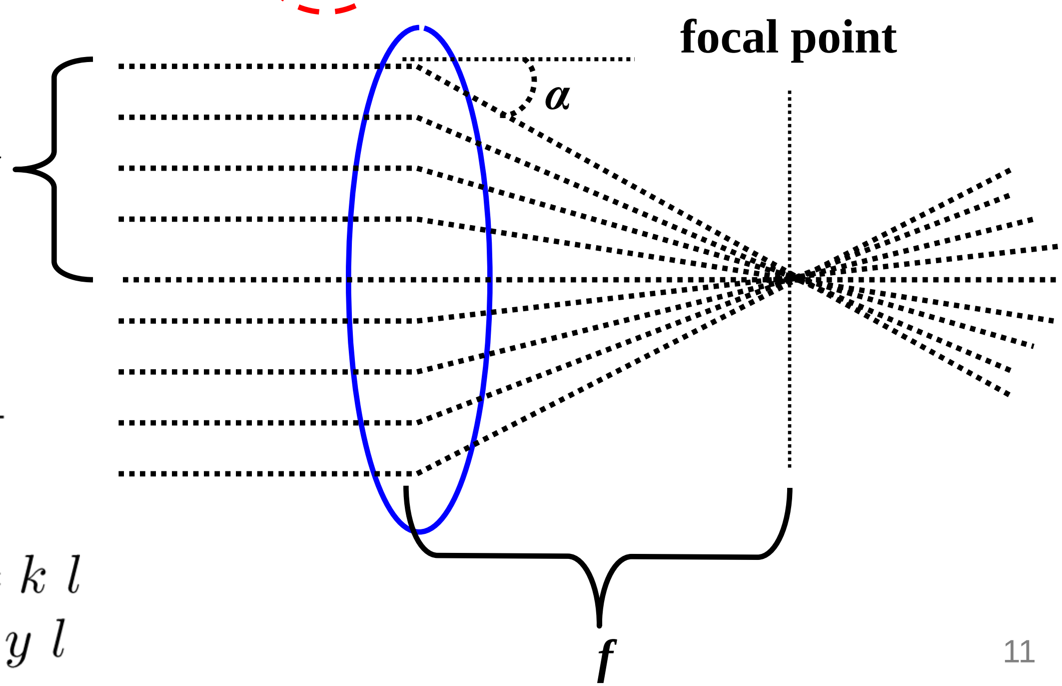
Focusing elements

- Magnetic element that deflects the beam by an angle proportional to the distance from its centre (equivalent to **ray optics**) provides focusing.
- The deflection angle is defined as $\alpha = -\frac{y}{f}$, for a lens of focal length ***f*** and small displacements ***y***.
- A magnetic element with length ***l*** and with a gradient ***g*** has a field $B_x = gy$ so that the deflection angle is

$$\alpha = -\frac{l}{\rho} = -\frac{q}{\beta E} B_x l = -\underbrace{\frac{q}{\beta E} g l}_{f^{-1}} y$$

- The normalised focusing strength is defined as

$$k = \frac{qg}{\beta E}$$

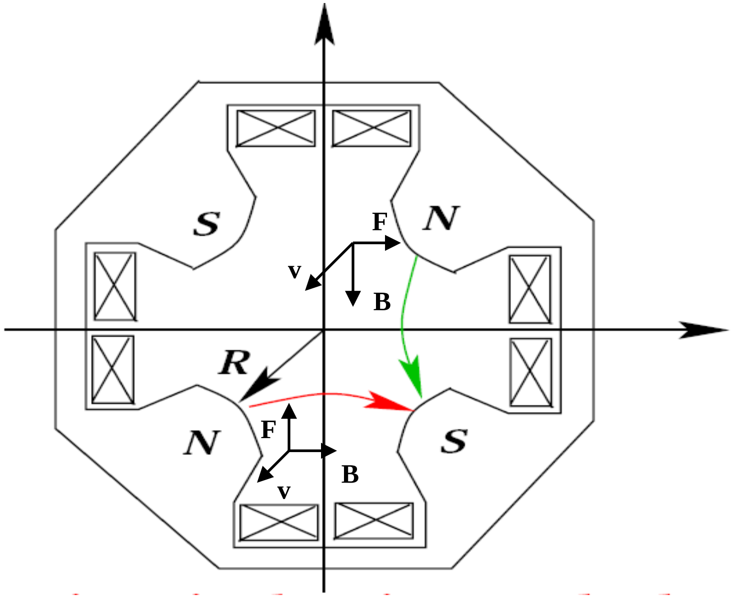


- In more practical units, for ***Z***=1

$$k[m^{-2}] = 0.2998 \frac{g[T/m]}{\beta E[GeV]}$$

- The focal length becomes $f^{-1} = k\ l$ and the deflection angle is $\alpha = -k\ y\ l$

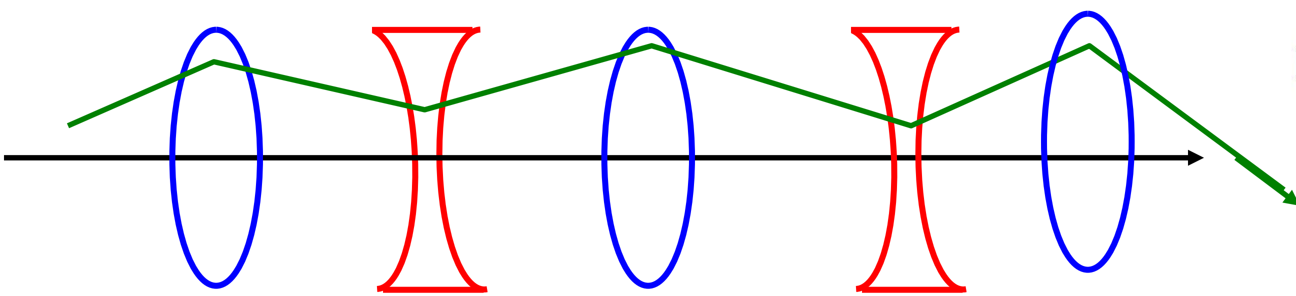
Quadrupoles

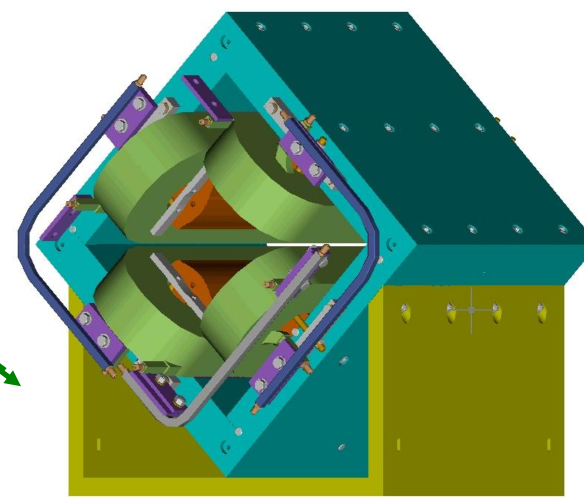


- Quadrupoles are focusing in one plane and defocusing in the other
- The field is $(B_x, B_y) = g(y, x)$
- The resulting force $(F_x, F_y) = k(y, -x)$
- Need to alternate focusing and defocusing in order to control the beam, i.e. **alternating gradient focusing**
- From optics we know that a combination of two lenses with focal lengths ***f₁*** and ***f₂*** separated by a distance ***d***

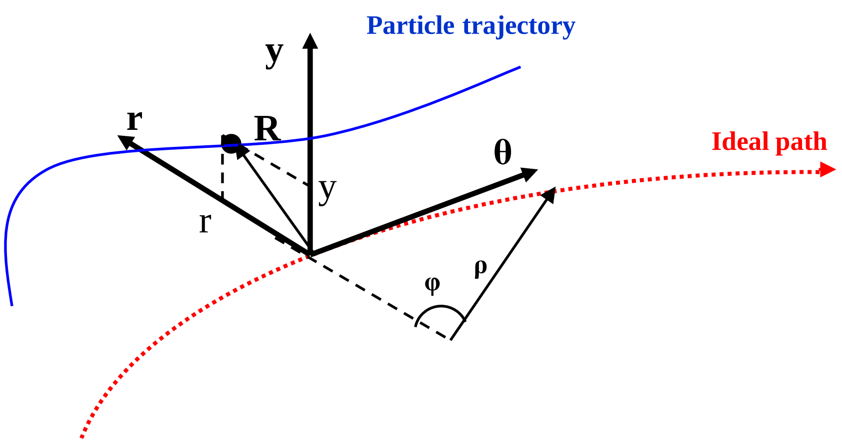
$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1\ f_2}$$

- If $f_1 = -f_2$, there is a **net focusing effect**, i.e. $\frac{1}{f} = \left|\frac{d}{f_1\ f_2}\right|$





Rotating coordinate system



- Consider a particle with charge ***q*** moving in the presence of transverse magnetic fields
- Choose cylindrical coordinate system ***(r,φ,y)***, with ***r = x+ρ*** and ***φ = s/ρ***
- The radius vector is

$$\mathbf{R} = \mathbf{R_0} + r\mathbf{u_r} + y\mathbf{u_y}$$

- For a small displacement ***dφ***

$$d\mathbf{u_r} = d\phi\mathbf{u}_\phi \ , \quad d\mathbf{u}_\phi = -d\phi\mathbf{u_r} \ , \quad d\mathbf{u_y} = 0$$

- Than the velocity is $\dot{\mathbf{R}} = \dot{r}\mathbf{u_r} + r\dot{\phi}\mathbf{u}_\phi + \dot{y}\mathbf{u_y}$
- And the acceleration $\ddot{\mathbf{R}} = (\ddot{r} - r\dot{\phi}^2)\mathbf{u_r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\mathbf{u}_\phi + \ddot{y}\mathbf{u_y}$
- Recall that the momentum time derivative is $\dot{\mathbf{p}} = \frac{d}{dt}(\gamma m_0\dot{\mathbf{R}}) = \gamma m_0\ddot{\mathbf{R}}$

Equations of motion

- Setting the electric field to zero and the magnetic field to be transverse

$$\mathbf{B} = (B_r, B_\phi, B_y) = (B_x, 0, B_y)$$

the Lorentz equations become

$$\dot{\mathbf{p}} = q\mathbf{v} \times \mathbf{B} = r\dot{\phi}\mathbf{u_r} + (\dot{y}B_x - \dot{r}B_y)\mathbf{u}_\phi - r\dot{\phi}B_x\mathbf{u_y}$$

- Replacing the momentum with the adequate expression and splitting the equations for the **r** and **y** direction

$$\begin{aligned} \gamma m_0(\ddot{r} - r\dot{\phi}^2) &= -qr\dot{\phi}B_y \\ \gamma m_0\ddot{y} &= qr\dot{\phi}B_x \end{aligned}$$

- Replace $r\dot{\phi} = v_\phi$, $r = x + \rho$ and as $v_\phi >> v_r$, $v_y \to P/v_\phi \approx \gamma m_0$
- The equations of motion in the new coordinates are

$$\begin{aligned} \frac{P}{v_\phi}(\ddot{x} - \frac{v_\phi^2}{\rho + x}) &= -qv_\phi B_y \\ \frac{P}{v_\phi}\ddot{y} &= qv_\phi B_x \end{aligned}$$

General equations of motion

- Note that for ***x<<ρ*** $\dfrac{1}{\rho+x} = \dfrac{1}{\rho}(1-\dfrac{x}{\rho})$
- It is convenient to consider the arc length ***s*** as the independent variable

$$ds = \rho d\phi = \rho\dot{\phi}dt = v_\phi \frac{\rho}{\rho+x}dt \approx v_\phi(1-\frac{x}{\rho})dt$$

and $$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_\phi(1-\frac{x}{\rho})\frac{d}{ds}\ , \quad \frac{d^2}{dt^2} \approx v_\phi^2\frac{d^2}{ds^2}$$

- Denote $\dfrac{dx}{ds} = x'\ ,\ \dfrac{d^2x}{ds^2} = x''$
- The general equations of motion are

$$\boxed{\begin{aligned} x'' &= \frac{1}{\rho}(1-\frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' &= \frac{qB_x}{P} \end{aligned}}$$

- **Remarks:**
  - Without the approximations, the equations are nonlinear and coupled!
  - The fields have to be defined

Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

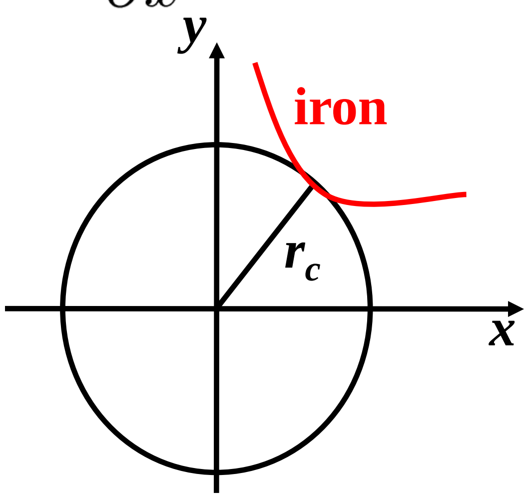
- Assuming a 2D field in ***x*** and ***y***, the vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe)

$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



There exist a complex potential of $z = x + iy$ with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Magnetic multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n$, $a_n = n\mu_n$ we have

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized units $b'_n = \frac{b_n}{10^{-4}B_0}r_0^{n-1}$, $a_n = \frac{a_n}{10^{-4}B_0}r_0^{n-1}$

on a reference radius, 10^{-4} of the main field to get

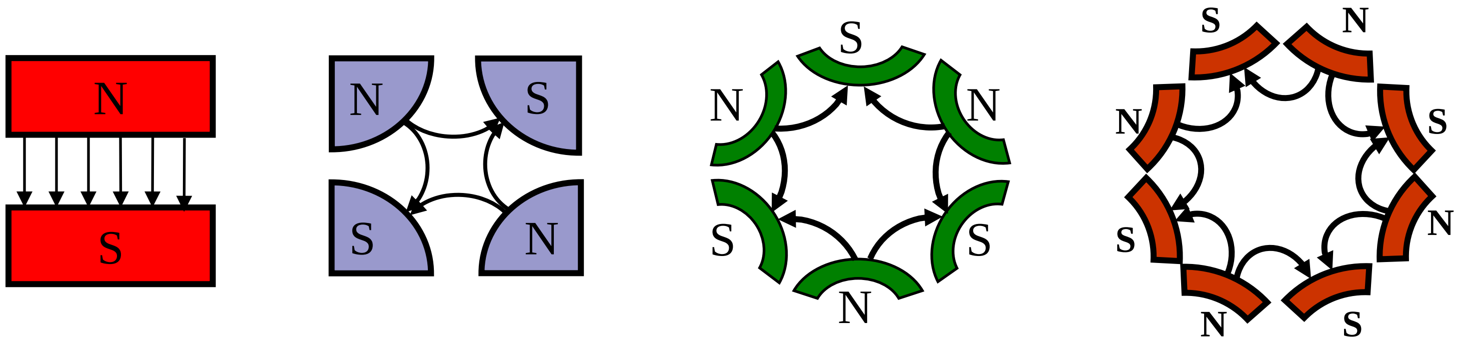
$$B_y + iB_x = 10^{-4}B_0\sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note**: ***n'=n-1*** is the US convention

Magnet definitions

- ***2n*-pole**:

dipole quadrupole sextupole octupole …



n: 1 2 3 4 …

- Normal: gap appears at the horizontal plane
- Skew: rotate around beam axis by $\boldsymbol{\pi/2n}$ angle
- Symmetry: rotating around beam axis by $\boldsymbol{\pi/n}$ angle, the field is reversed (polarity flipped)

Parameters

- Field needed to bend a particle and force it into a trajectory

$$\vec{F} = q(\vec{v} \times \vec{B}) = m\vec{a} \longrightarrow \begin{array}{l} q\,v\,B = mv^2/R \\ B\,R = mv/q \end{array}$$

BR = Br = p/0.2998 for an electron or a proton when *p* in GeV

- Particle species and energy define the accelerator
- The first decision in the design of a circular accelerator is the trade-off between tunnel circumference and magnetic field.
  - For iron magnets $\boldsymbol{B} \leq 2T$
  - For ***B*** > 2T superconducting technology needed
  - Size of the beam and optics define the quadrupole fields. Strength usually not as stringent, apart in special areas where extremely strong focusing is needed (e.g. interaction regions of colliders)
- Field should be linear to the 10^{-4} level. **Transfer function, i.e** integrated magnet strength divided by the current needs to be constant.

Magnetic materials

- <u>Reminder:</u> $\boldsymbol{M} = \chi \boldsymbol{H}$, $\boldsymbol{B} = m\boldsymbol{H} = m_0(\boldsymbol{H}+\boldsymbol{M}) = m_0\,(1+\chi)\,\boldsymbol{H}$
- where, ***M*** Magnetization, ***H*** excitation field, ***B*** Magnetic induction, *m* permeability, χ susceptibility
- Most materials present **diamagnetism** ($1+\chi>1$, i.e. field strengthens) or **paramagnetism** ($1+\chi<1$, i.e. field weakens) . In both cases, the magnetization disappears when the external excitation field is zero
- Special materials show **ferromagnetism**, i.e. minuscule magnetic domains exist in the material and get aligned in the presence of a magnetic field.
- **Saturation** (material cannot absorb stronger magnetic field) is reached asymptotically at 0 ºK with no thermal disorder when all the domains are aligned. Above the **Curie temperature** material looses its ferromagnetic behavior

	$\mu_0 M_{sat}$ [Tesla @ 0 K]	Curie Temp [C]
Iron	2.18	770
Nickel	0.64	358
Cobalt	1.81	1120

Magnetic potential

The Maxwell laws for magnetostatics:

$$\nabla . \mathrm{B} = 0; \nabla \times \mathrm{H} = \mathrm{j}.$$

In the absence of currents j=0 and B can be expressed as the gradient of a scalar potential

$$\mathrm{B} = \nabla V \quad \text{and } \nabla^2 V = 0$$

The **Laplace equation** has a solution in cylindrical coordinates of the form:

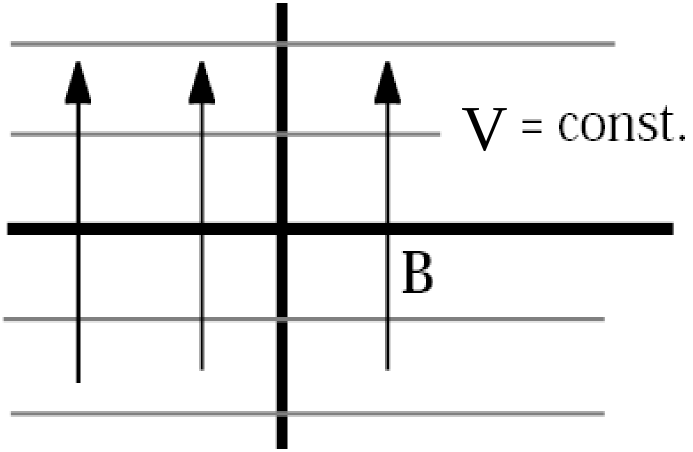
$$V = \sum_{\mathrm{n}} \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \right)$$

where a_n and b_n correspond to pure skew and normal components

Ideal pole shape

- In iron dominated magnets, the **field** is determined by the **shape** of the steel poles (lines of constant scalar potential).
- Conductor coils are wounded around the poles to create the magnetic field.
- Due to boundary conditions, pole surfaces are **equipotential lines**

Dipole (n = 1)



$$V(r,\theta) = a_1 r\cos\theta + b_1 r\sin\theta$$

$$V(x,y) = a_1 x + b_1 y$$

$$\mathrm{B}_x = a_1$$

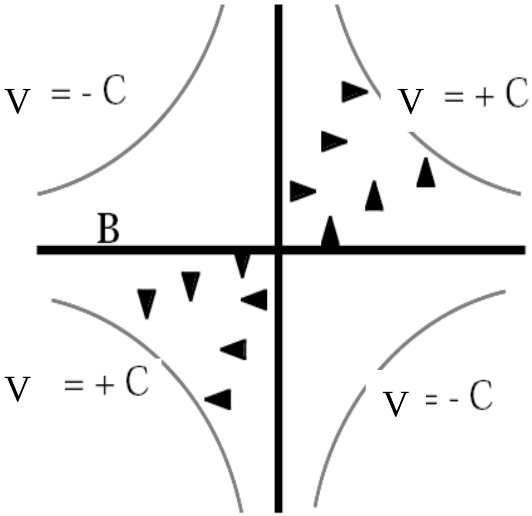
$$\mathrm{B}_y = b_1$$

$$\text{Pole}: y = \pm d/2$$

with d the gap size

Ideal pole shape

Quadrupole (n = 2)



$$V(r,\theta) = a_2 r^2 \cos 2\theta + b_2 r^2 \sin 2\theta$$

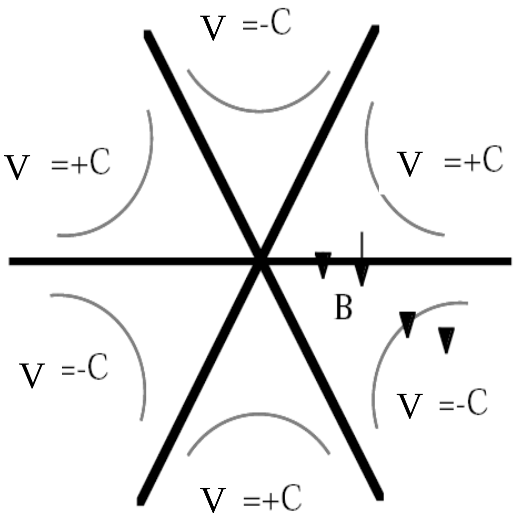
$$V(x,y) = a_2(x^2 - y^2) + 2b_2 xy$$

$$\mathrm{B}_x = 2(a_2 x + b_2 y)$$

$$\mathrm{B}_y = 2(-a_2 y + b_2 x)$$

$$\mathrm{Pole}: xy = \pm R^2/2$$

Sextupole (n = 3)



$$V(r,\theta) = a_3 r^3 \cos 3\theta + b_3 r^3 \sin 3\theta$$

$$V(x,y) = a_3(x^3 - 3y^2 x) + b_3(3yx^2 - y^3)$$

$$\mathrm{B}_x = 3a_3(x^2 - y^2) + 6b_3 xy$$

$$\mathrm{B}_y = -6a_3 xy + 3b_3(x^2 - y^2)$$

$$\mathrm{Pole}: 3x^2 y - y = \pm R^3$$

with R the gap radius

Magnets with normal symmetries

- **Dipole**: $$V(\theta) = -V(\pi + \theta)$$ $$V(\theta) = V(2\pi + \theta)$$

only **b**$_n$ with n=1,3,5, etc.

- **Quadrupole**: $$\begin{cases} V(\theta) = -V(\pi/2 + \theta) \\ V(\theta) = -V(3\pi/2 + \theta) \end{cases}$$ $$\begin{cases} V(\theta) = V(\pi + \theta) \\ V(\theta) = V(2\pi + \theta) \end{cases}$$

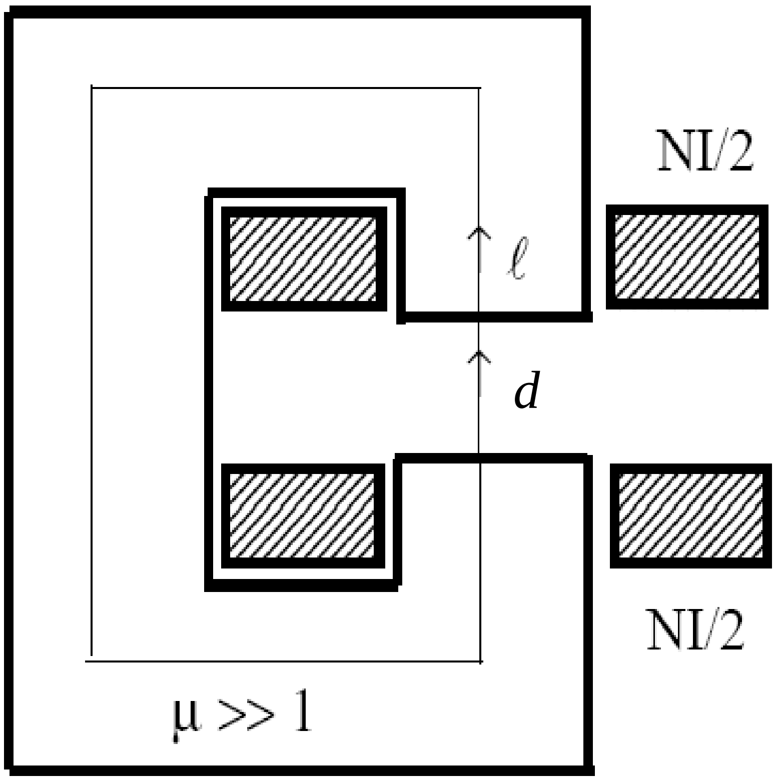
only **b**$_n$ with n=2,6,10, etc.

- **Sextupole**: $$\begin{cases} V(\theta) = -V(\pi/3 + \theta) \\ V(\theta) = -V(\pi + \theta) \\ V(\theta) = -V(5\pi/3 + \theta) \end{cases}$$ $$\begin{cases} V(\theta) = V(2\pi/3 + \theta) \\ V(\theta) = V(4\pi + \theta) \\ V(\theta) = V(2\pi + \theta) \end{cases}$$

only **b**$_n$ with n=3,9,15, etc.

- In practice poles have a finite width
- Impose the same symmetries to the pole constrains the harmonics
- Longitudinally, the magnet end will generate non linear fringe fields ➔ **shims**
- Sharp edges will cause saturation and the field becomes non linear ➔ **chamfered** ends
- Numerical methods applied to define the shape and machined with high precision.

Dipole current



- **B** is constant around the loop l and in the gap
- According to **Ampere's law**

$$\oint \vec{H}\cdot d\vec{s} = \int_A \vec{J}\cdot d\vec{A} \Rightarrow \begin{array}{l} H_{iron}\, l + H_{gap}\, d = NI \\ \cancel{\frac{1}{\mu_o \mu} B_{iron}\, l} + \frac{1}{\mu_0} B\, d = NI \end{array}$$

$\mu \gg 1$

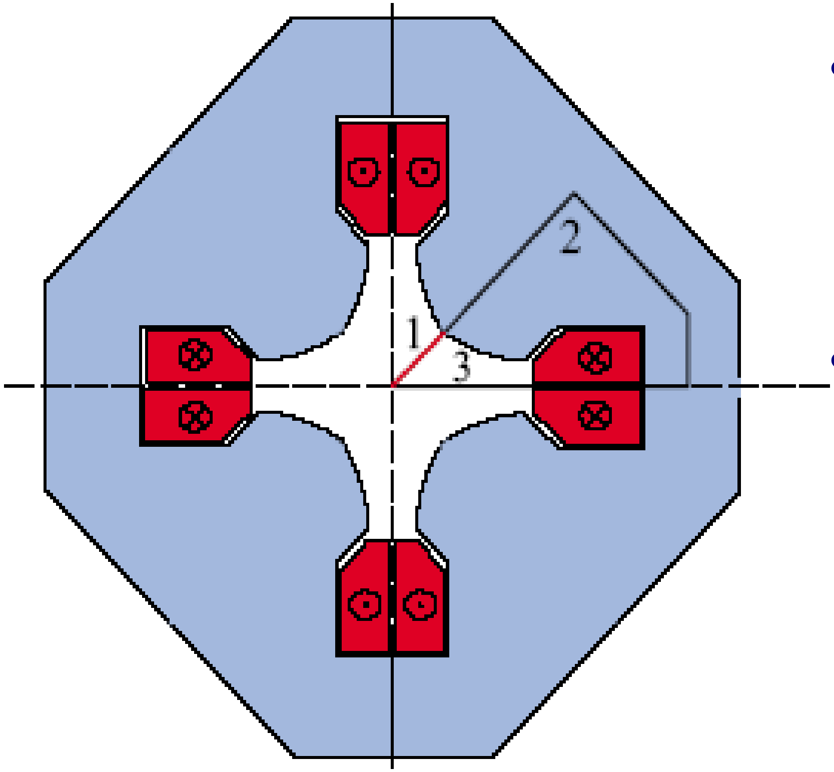
with N the number of turns

$$\boxed{B = \frac{\mu_0 NI}{d}}$$

- The number of conductors and current density can be optimized
- Limited by the iron saturation

$B \leq 2T$

Quadrupole current



- **B** is perpendicular to d**s** along path 3 and within the iron (path 2) μ>>1
- Only non-zero component along path 1 and using again Ampere's law

$$\oint \vec{H} \cdot d\vec{s} = \int_1 \mathrm{H}_1 \cdot ds = NI$$

- The gradient G in a quadrupole is

$$B_x = Gy; \quad B_y = Gx$$

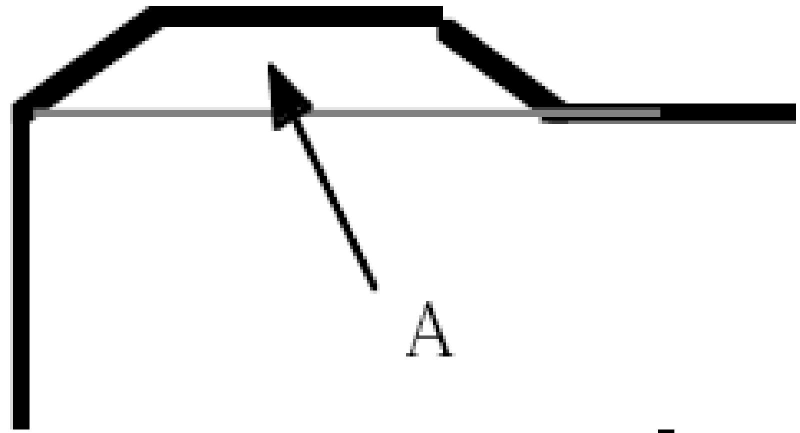
$$\mathrm{H}_1 = \frac{G}{\mu_0} \sqrt{x^2 + y^2} = \frac{G}{\mu_0} r$$

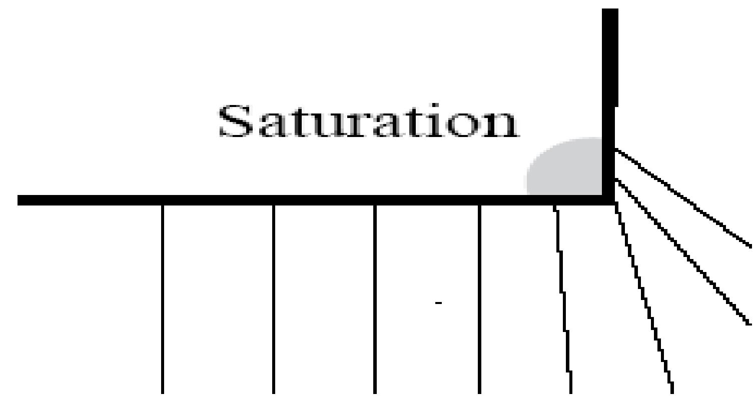
and after integrating

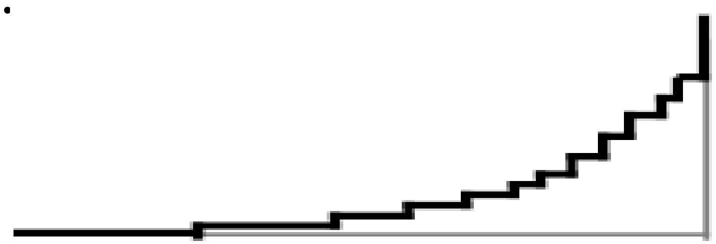
$$\boxed{G = \frac{2\mu_0 NI}{r_0^2}}$$

Practical problems

- The finite pole width creates errors.
- To compensate, small steps are added at the outer ends of each of the pole called **shims**.
- Saturation at the corners at the longitudinal end also creates high order multipoles
- **Chamfering** is performed, the size of which is estimated using numerical calculations
- Eddy currents oppose to the change of magnetic field and produce losses
- Eddy currents are minimized by **transposing** the conductors in the coil.
- Eddy currents in the yoke are avoided using **laminations**

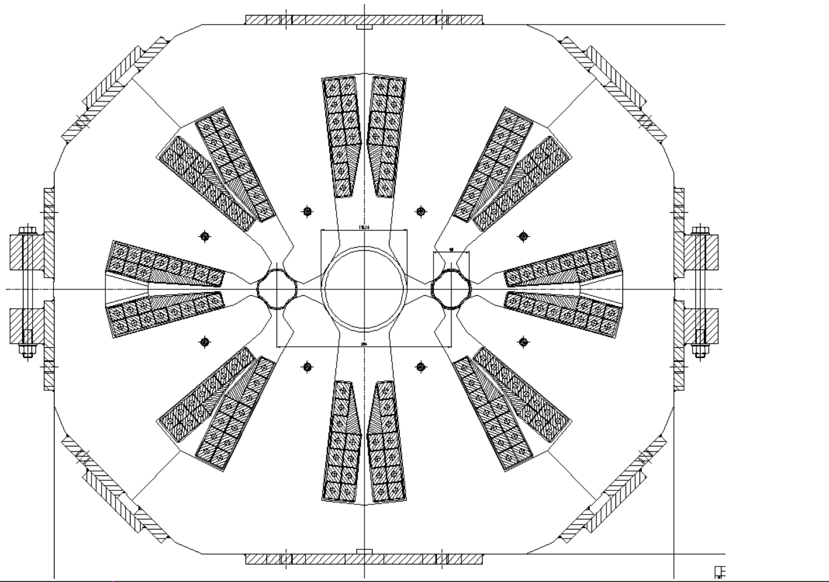


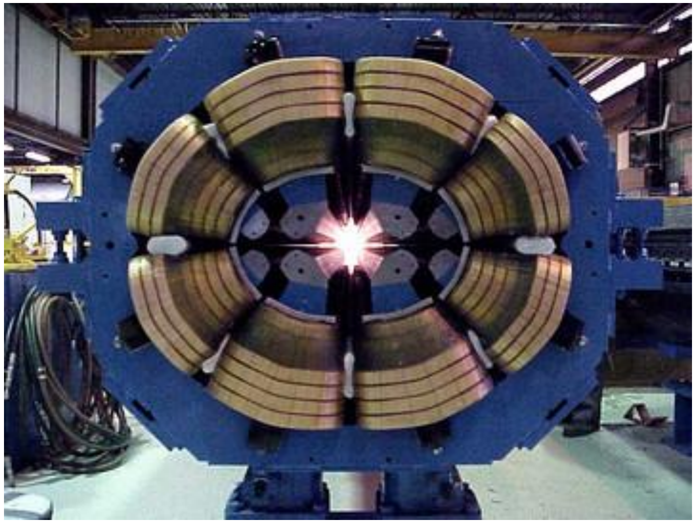


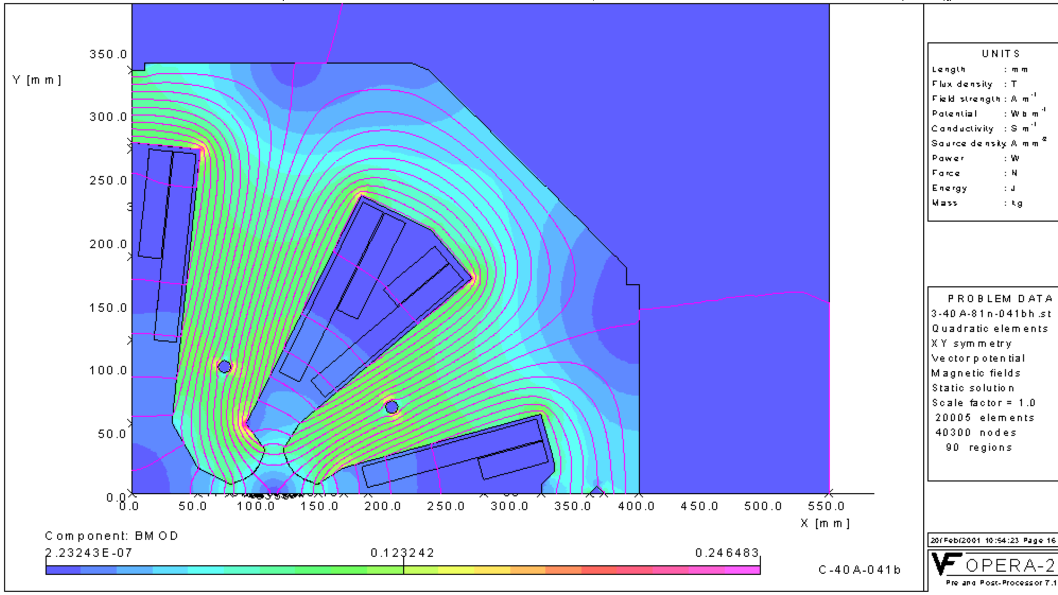


Example: MQWA quadrupole

- Twin aperture quadrupole for LHC built in collaboration with TRIUMF by Alstom, CANADA.
- Pole geometry is identical in the nose
- Disposition of coils around the poles is non symmetric.
- Field calculations require numerical methods done with Opera.







Equations of motion – Linear fields

- Equations of motion (see slide 15)

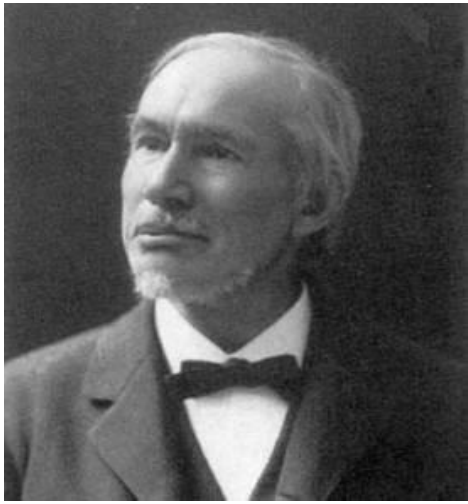
$$\left\{\begin{array}{rcl} x'' & = & \frac{1}{\rho}(1-\frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' & = & \frac{qB_x}{P} \end{array}\right.$$

- Consider s-dependent fields from dipoles and normal quadrupoles $B_y = B_0(s) - G(s)x\ , \quad B_x = -G(s)y$
- The total momentum can be written $P = P_0(1+\frac{\Delta P}{P})$
- The magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and the normalized gradient $k = \frac{G}{B_0\rho}$
- The equations become

$$\boxed{\begin{array}{rcl} x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right)x & = & \frac{1}{\rho(s)}\frac{\Delta P}{P} \\ y'' + k(s)\ y & = & 0 \end{array}}$$

- Inhomogeneous equations with *s*-dependent coefficients
- Note that the term ***1/ρ²*** corresponds to the **week focusing**
- The term ***ΔP/(Pρ)*** is non-zero for off-momentum particles

Hill's equations



George Hill

- Solutions are a combination of the ones from the homogeneous and inhomogeneous equations
- Consider first particles with the design momentum. The equations of motion become

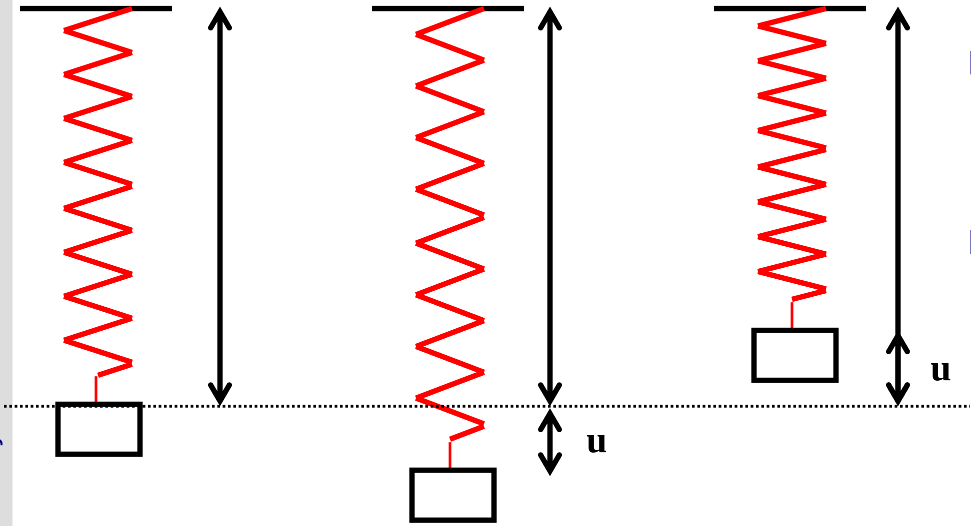
$$\boxed{\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}}$$

with $K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right)\ ,\ \ K_y(s) = k(s)$

Hill's equations of linear transverse particle motion

- Linear equations with *s*-dependent coefficients (**harmonic oscillator** with "time" dependent frequency)
- In a ring or in transport line with symmetries, coefficients are periodic $K_x(s) = K_x(s+C)\ ,\ \ K_y(s) = K_y(s+C)$
- Not practical to get analytical solutions for all accelerator

Harmonic oscillator – spring



- Consider $K(s) = k_0$ = constant

$$u'' + k_0\ u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\ u(0) + S(s)\ u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0}s)\ , \quad S(s) = \frac{1}{\sqrt{k_0}}\sin(\sqrt{k_0}s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|}s)\ , \quad S(s) = \frac{1}{\sqrt{|k_0|}}\sinh(\sqrt{|k_0|}s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

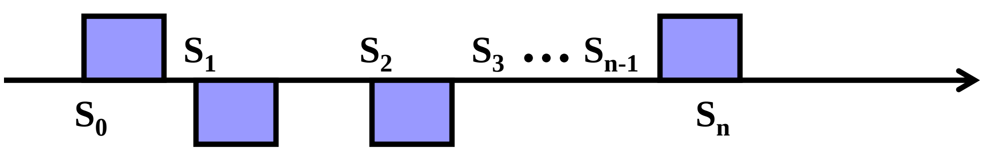
- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$

  which is always true for conservative systems

- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- The accelerator can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \underbrace{\mathcal{M}(s_1|s0)}_{\text{from } s_0 \text{ to } s_1}$$



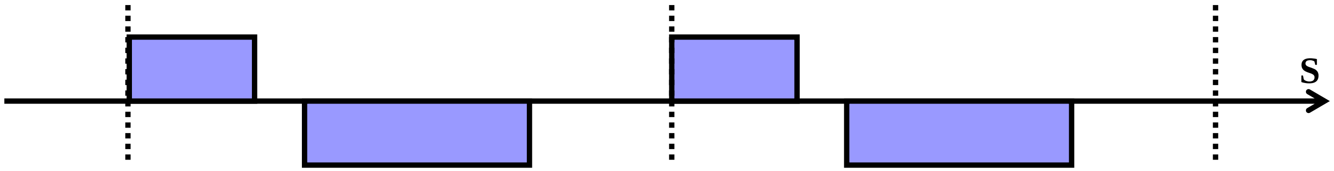
from $\boldsymbol{s_0}$ to $\boldsymbol{s_2}$

from $\boldsymbol{s_0}$ to $\boldsymbol{s_3}$

from $\boldsymbol{s_0}$ to $\boldsymbol{s_n}$

Symmetric lines

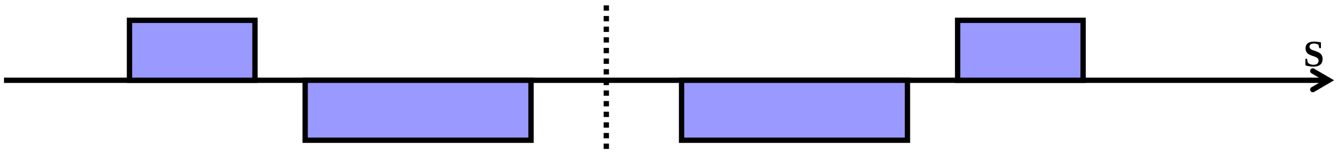
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}\cdot\mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r\cdot\mathcal{M} = \begin{pmatrix} ad+bc & 2bd \\ 2ac & ad+cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Uncoupled motion

Transfer matrix of a drift

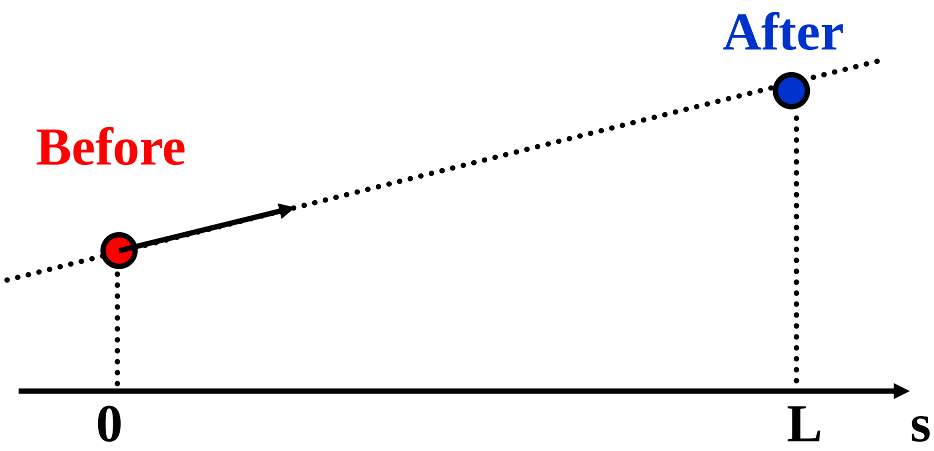
- Consider a drift (no magnetic elements) of length ***L=s-s₀***

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

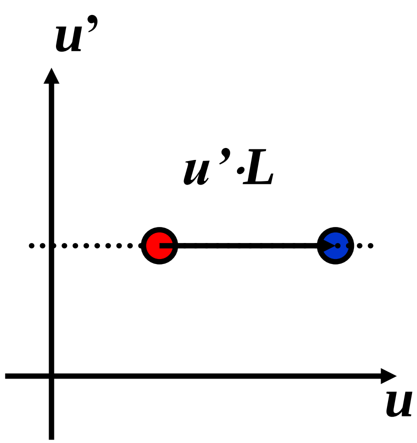
$$\boxed{\mathcal{M}_{\text{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{aligned} u(s) &= u_0 + \overbrace{(s-s_0)}^{L} u'_0 = u_0 + L u'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if particle has a slope which remains unchanged.



Real Space



Phase Space

Focusing - defocusing thin lens

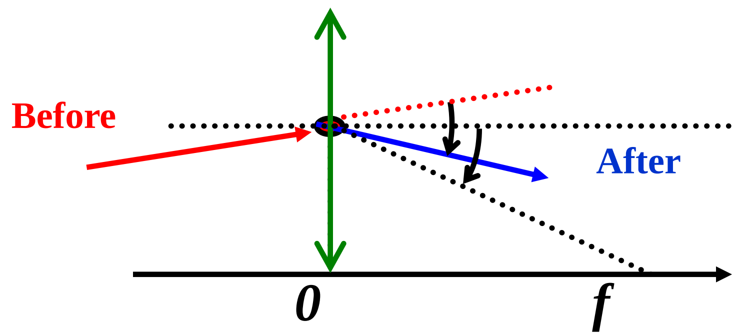
- Consider a lens with focal length ***f***

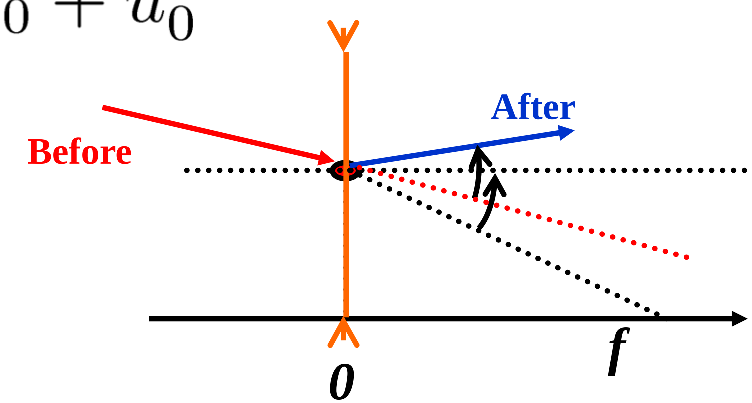
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

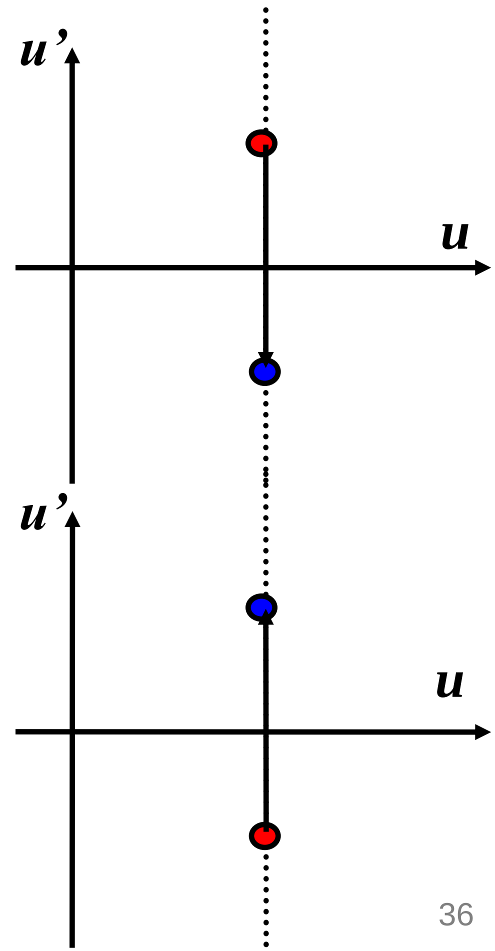
$$\boxed{\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}}$$

- Slope **diminishes** (focusing) or **increases** (defocusing) for positive position, which remains unchanged.

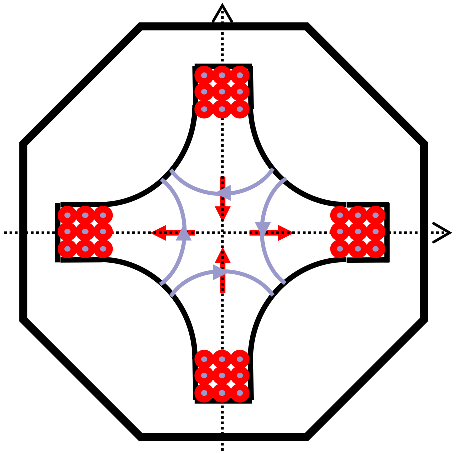
$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= \mp\frac{1}{f}u_0 + u'_0 \end{aligned}$$







Quadrupole



- Consider a quadrupole magnet of length ***L***. The field is

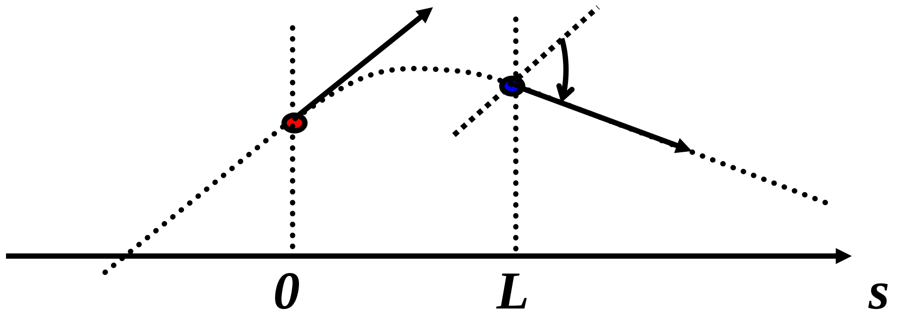
$$B_y = -G(s)x \ , \quad B_x = -G(s)y$$

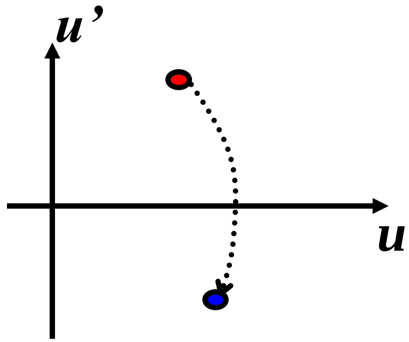
with normalized quadrupole gradient (in $\mathbf{m}^{-2}$)

$$k = \frac{G}{B_0 \rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$





Quadrupole II

- For a focusing quad (***k*>0**)

$$\mathcal{M}_{\mathrm{QF}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

- For a defocusing quad (***k*<0**)

$$\boxed{\mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} \cosh(\sqrt{|k|}L) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}L) \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix}}$$

- By setting $\sqrt{k}L \to 0$

$$\mathcal{M}_{\mathrm{QF,QD}} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \mathcal{M}_{\mathrm{lens}}$$

- **Note** that the **sign** of *k* or *f* is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Sector Dipole

- Consider a dipole of length ***L***. By setting in the focusing quadrupole matrix

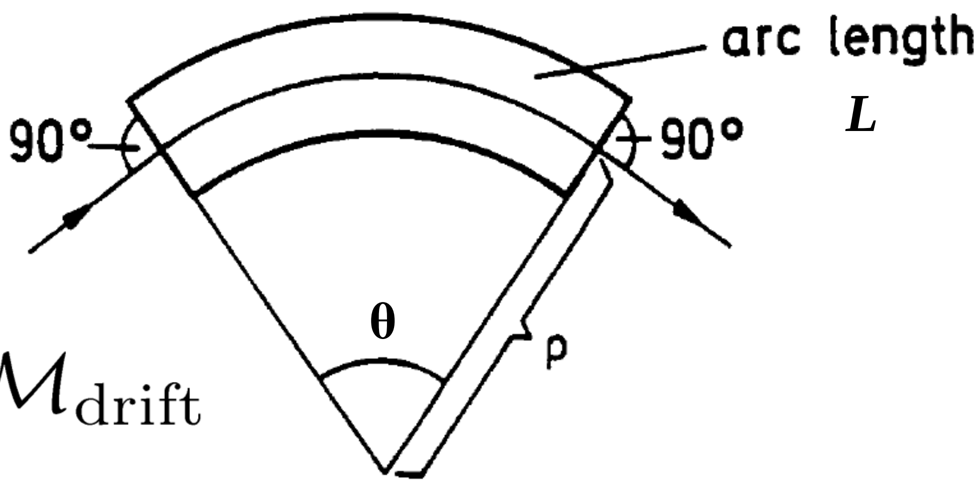
$$k = \frac{1}{\rho^2} > 0$$

the transfer matrix for a sector dipole becomes

$$\boxed{\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}}$$

with a bending radius $\theta = \frac{L}{\rho}$

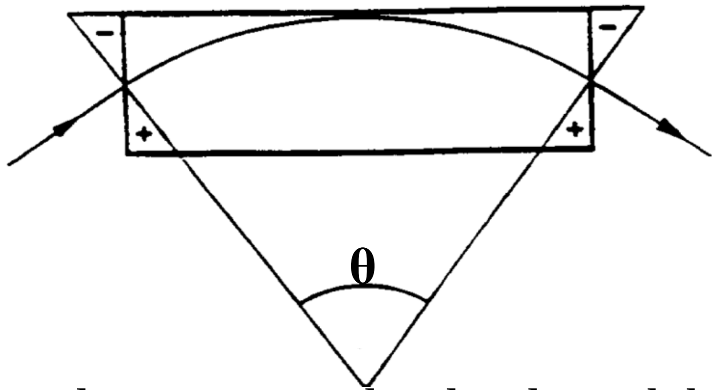
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

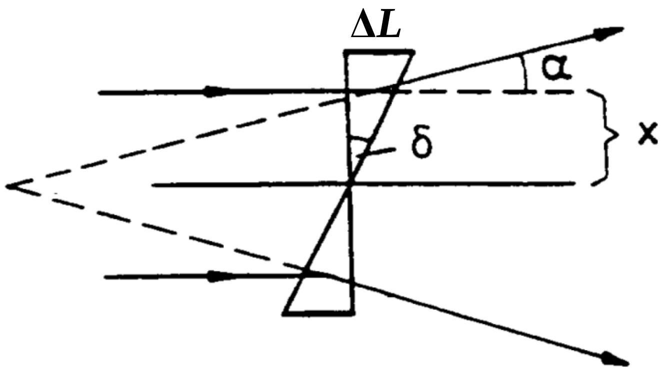


$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, **edge focusing** appears in the vertical plane
- Matrix can be generalized by adding strong focusing for **synchrotron** magnets

Rectangular Dipole





- Consider a rectangular dipole with bending angle θ. At each edge of length ΔL, the deflecting angle is changed by

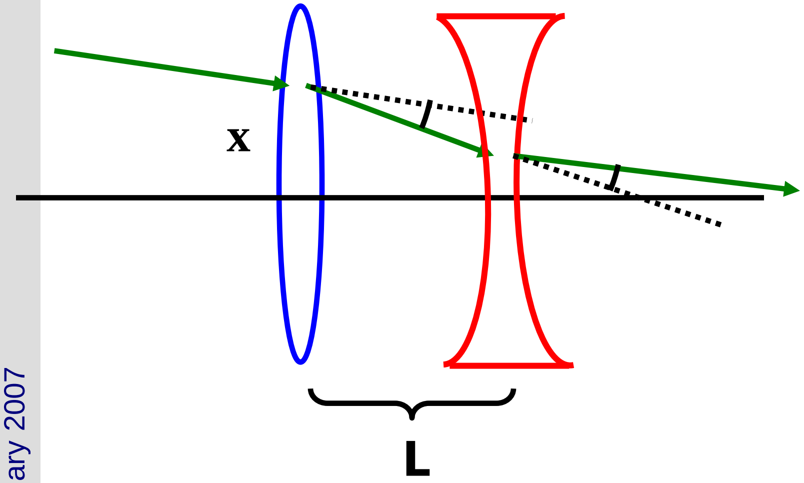
$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$

i.e., it acts as a thin defocusing lens with focal length $\frac{1}{f} = \frac{\tan\delta}{\rho}$

- The transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$ with $\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ -\frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$
- For **θ<<1, δ=θ/2**
- In deflecting plane (like **drift**) in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Quadrupole doublet



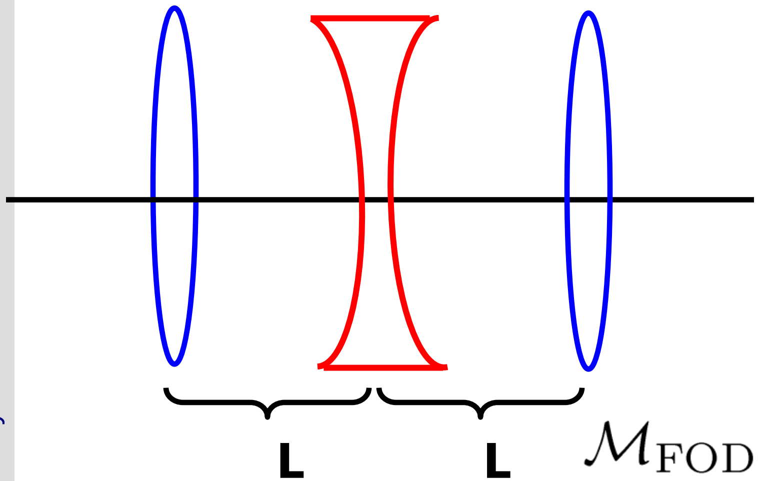
- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths f_1 and f_2 separated by a distance L.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1-\frac{L}{f_2} \end{pmatrix}$$

with the **total focal length** $$\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\ f_2}$$

- Setting $f_1 = -f_2 = f$ $$\frac{1}{f^\star} = \frac{L}{f^2}$$
- **Alternating gradient focusing** seems overall focusing
- This is only valid in thin lens approximation

FODO Cell



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths *f*.
- The symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

with the transfer matrices

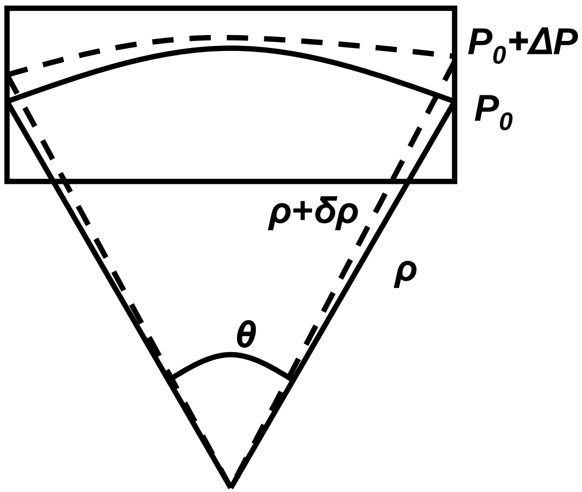
$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ \frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum P_0
- What happens for off-momentum particles, i.e. particles with momentum $P_0+\Delta P$?
- Consider a dipole with field B and bending radius ρ
- Recall that the magnetic rigidity is $B\rho = \frac{P_0}{q}$ and for off-momentum particles



$$B(\rho+\Delta\rho) = \frac{P_0+\Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Considering the effective length of the dipole unchanged

$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

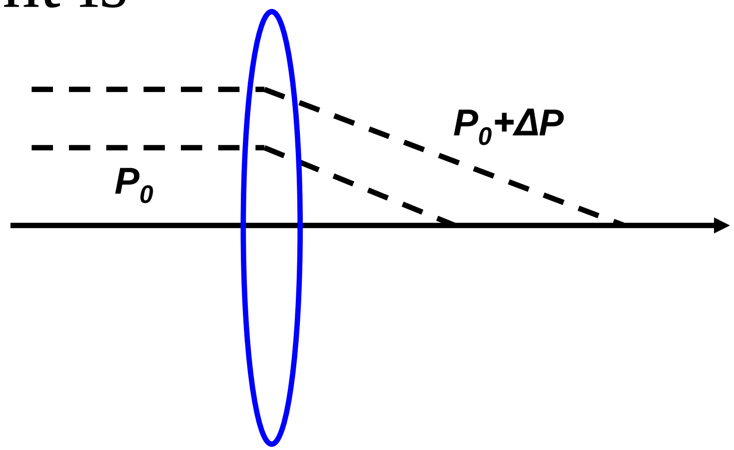
- Off-momentum particles get different deflection (different orbit)

$$\boxed{\Delta\theta = -\theta\frac{\Delta P}{P_0}}$$

Off-momentum particles and quadrupoles

- Consider a quadrupole with gradient ***G***
- Recall that the normalized gradient is

$$K = \frac{q\,G}{P_0}$$



and for off-momentum particles

$$\Delta K = \frac{dK}{dP}\Delta P = -\frac{qG}{P_0}\frac{\Delta P}{P_0}$$

- Off-momentum particle gets different focusing

$$\Delta K = -K\frac{\Delta P}{P_0}$$

- This is equivalent to the effect of **optical lenses** on **light of different wavelengths**

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way, the equations of motion are split in two parts

$$\begin{aligned} x''_H + K_x(s)x_H &= 0 \\ x''_I + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- The **dispersion function** can be defined as $D(s) = \dfrac{x_I(s)}{\Delta P/P}$
- The dispersion equation is

$$\boxed{D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}}$$

Dispersion solution for a bend

- Simple solution by considering motion through a sector dipole with constant bending radius ***ρ***
- The dispersion equation becomes $D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$
- The solution of the homogeneous is harmonic with frequency ***1/ρ***
- A particular solution for the inhomogeneous is $D_p = \text{constant}$ and we get by replacing $D_p = \rho$
- Setting ***D(0) = D₀*** and ***D'(0) = D'₀***, the solutions for dispersion are

$$\begin{aligned} D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0' \rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\ D'(s) &= -\frac{D_0}{\rho} \sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho}) \end{aligned}$$

General dispersion solution

- General solution possible with perturbation theory and use of Green functions
- For a general matrix $\mathcal{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S(s)' \end{pmatrix}$ the solution is

$$D(s) = S(s)\int_{s_0}^{s} \frac{C(\bar{s})}{\rho(\bar{s})} d\bar{s} + C(s)\int_{s_0}^{s} \frac{S(\bar{s})}{\rho(\bar{s})} d\bar{s}$$

- One can verify that this solution indeed satisfies the differential equation of the dispersion (and the sector bend)
- The general betatron solutions can be obtained by 3X3 transfer matrices including dispersion

$$\mathcal{M}_{3\times 3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

- Recalling that $x(s) = x_B(s) + D(s)\dfrac{\Delta P}{P}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix}$$ and $$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

3x3 transfer matrices - Drift, quad and sector bend

- For **drifts** and **quadrupoles** which do not create dispersion the 3x3 transfer matrices are just

$$\mathcal{M}_{\text{drift,quad}} = \begin{pmatrix} \mathcal{M}_{2\times 2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

- For the deflecting plane of a **sector bend** we have seen that the matrix is

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and in the non-deflecting plane is just a drift.

3x3 transfer matrices - Synchrotron magnet

- Synchrotron magnets have focusing and bending included in their body. From the solution of the sector bend, by replacing ***1/ρ*** with ***K=(1/ρ² - k)^{1/2}***.
- For ***K*>0** $\mathcal{M}_{\mathrm{syF}} = \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sqrt{K}} & \frac{1-\cos\psi}{\rho K} \\ -\sqrt{K}\sin\psi & \cos\psi & \frac{\sin\psi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$
- For ***K*<0** $\mathcal{M}_{\mathrm{syD}} = \begin{pmatrix} \cosh\psi & \frac{\sinh\psi}{\sqrt{|K|}} & -\frac{1-\cosh\psi}{\rho|K|} \\ \sqrt{|K|}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$

with $\psi = \sqrt{|k + \frac{1}{\rho^2}|}l$

3x3 transfer matrices - Rectangular magnet

- The end field of a rectangular magnet is simply the one of a quadrupole. The transfer matrix for the edges is

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan(\theta/2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

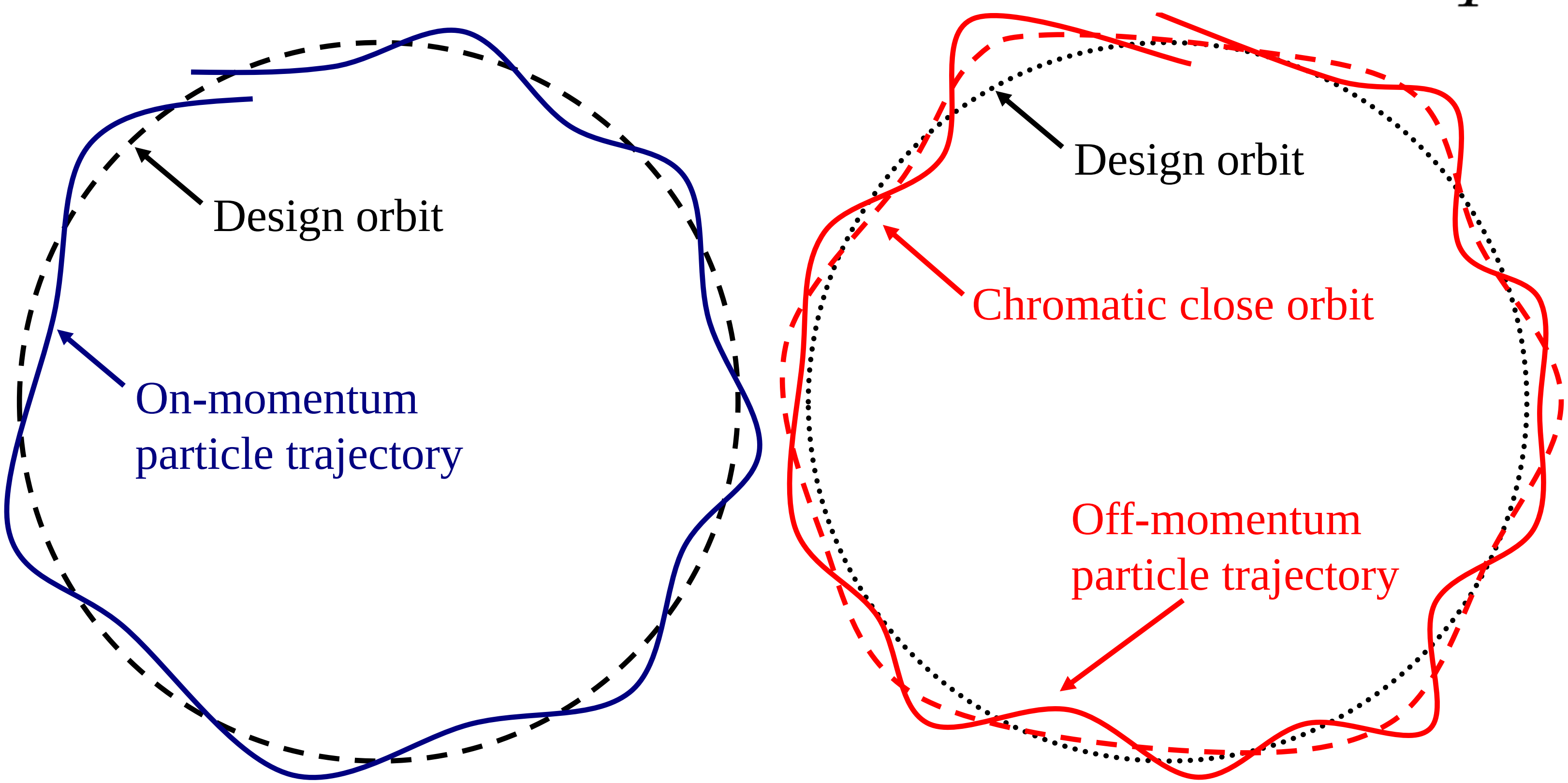
- The transfer matrix for the body of the magnet is like the sector
- The total transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sect}} \cdot \mathcal{M}_{\text{edge}}$

$$\mathcal{M}_{\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan(\theta/2) \\ 0 & 0 & 1 \end{pmatrix}$$

Chromatic closed orbit

- Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit
- Distance from the design orbit depends linearly with momentum spread and dispersion

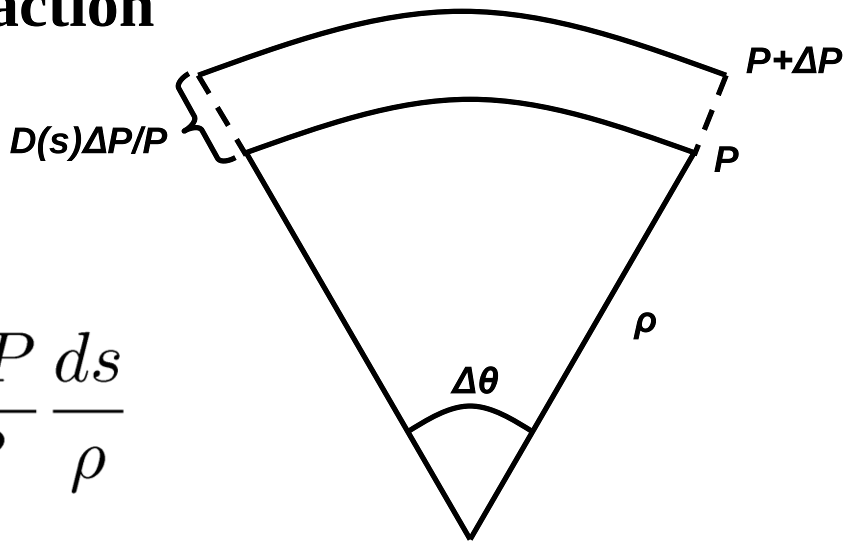
$$x_D = D(s)\frac{\Delta P}{P}$$



Momentum compaction

- Off-momentum particles on the dispersion orbit travel in a different path length than on-momentum particles
- The change of the path length with respect to the momentum spread is called **momentum compaction**

$$\alpha_c = \frac{\Delta C}{C} \bigg/ \frac{\Delta P}{P}$$



- The change of circumference is

$$\Delta C = \oint D \frac{\Delta P}{P} d\theta = \oint D \frac{\Delta P}{P} \frac{ds}{\rho}$$

- So the momentum compaction is

$$\boxed{\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds = \left\langle \frac{D(s)}{\rho(s)} \right\rangle}$$

Transition energy

- The revolution frequency of a particle is $f = \frac{v}{2\pi\rho} = \frac{\beta c}{2\pi\rho}$
- The change in frequency is $\frac{\Delta f}{f} = \frac{\Delta\rho}{\rho} - \frac{\Delta\beta}{\beta}$
- From the relativistic momentum $Pc = \beta E$ we have

$$\frac{\Delta P}{P} = \frac{\Delta\beta}{\beta} + \frac{\Delta E}{E} \rightarrow \beta^2 \frac{\Delta P}{P}$$

for which $\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta P}{P}$ and the revolution frequency

$\frac{\Delta f}{f} = (\frac{1}{\gamma} - \alpha_c)\frac{\Delta P}{P}$. The transition energy is defined by

$$\boxed{\gamma_t = \frac{1}{\sqrt{\alpha_c}}}$$

Problems I

1) Consider an **electron** storage ring with **circumference** of **850m** and **energy** of **6GeV**. If the maximum **bending field** available is **0.854T**, what is the percentage of the circumference occupied by dipoles? With dipoles of **2.3m** long, find the integrated dipole strength, the bending angle and the number of dipoles.

2) In the SNS ring, the necessary bending of the proton beam in the arcs is provided by **32 dipoles** with a **17cm gap** and a **magnetic length** of **1.5m**. If the coil is built with **20 turns**, how much should the power supply current be in order to give the necessary field for **1GeV** and **1.3 GeV** operation. The **rest energy** of the proton is **938.273 MeV**.

3) Trace the poles of a decapole and dodecapole magnet. What is the angle between the centre of each pole in each case? Derive this angle for general ***2n***-pole magnets?

4) Use the expansion of the scalar potential in polar coordinates in order to show that the potential is symmetric by a rotation of $\boldsymbol{\pi}$. Prove that the first allowed multi-pole for a normal quadrupole magnet is a 12-pole ($\mathbf{b}_6$), the second a 20-pole ($\mathbf{b}_{10}$), etc. Is there a general rule for all multi-pole magnets?

Problems II

5) Prove that the transfer matrices of two symmetric cells and of one cell with mirror symmetry have their determinant equal to 1. Derive the transfer matrix of a particle moving in the opposite direction in the two above cases.

6) Find the focal length of a thin focusing and defocusing quadrupole. To do so, consider an incoming parallel beam (in ***x*** or in ***y*** depending on the quad) and propagate it using the quad and a drift, and find the drift length in order to get **0** displacement. Do the same for both planes for a doublet formed by the two quads, with distance ***l*** between them.

7) Write the transfer matrix of a FODO cell for which the integrated focal length is ***f* = *2f*** and the drift has distance ***l***. For numerical evaluation you will need that **Bρ = 26.68**, the quad length **0.509m** the quadrupole gradient **12T/m** and the distance between quads **6.545m**.