Basic definitions and formulas

Y. Papaphilippou and N.Catalan-Lasheras

USPAS, Cornell University, Ithaca, NY
20th June – 1st July 2005

Outline

- Units and dimensions
- A crash course in relativity
  - Historical background
  - Lorentz transformations (length contraction and time dilation)
  - 4-vectors and Einstein's relation
  - Conservation laws and Particle collisions
- Concepts of Electromagnetism
  - Maxwell's equations
  - Lorentz force and particle dynamics
  - Electromagnetic waves

Unit systems

- Metric MKS or SI system (meter, kilogram, second)
- Gaussian cgs system (centimeter, gram, second) using electrostatic system of units (esu) or electromagnetic system of units (emu)
- Conversion between SI and cgs (emu):
  - Force: 1 dyne [g cm s^{-2}] = 10^{-5} Newton [kg m s^{-2}]
  - Energy: 1 erg [g cm^2 s^{-2}] = 10^{-7} Joule [kg m^2 s^{-2}]
  - Power : 1 g cm^2 s^{-3} = 10^{-7} Watt [kg m^2 s^{-3}]
  - Electric charge: 1abCoulomb = 10 Coulomb
  - Electric current: 1abAmpere = 0.1 Ampere
  - Electric potential: 1abVolt [g cm^2 s^{-3} abA^{-1}] = 10^{-8} Volt [kg m^2 s^{-3} A^{-1}]
  - Electric field: 1abVolt/cm = 10^{-6} Volt/meter
  - Magnetic induction: 1 Gauss [g s^2 abA^{-1}] = 10^{-4} Tesla [kg s^2 A^{-1}]
- Converting from emu to esu by replacing ab prefix with stat and introduce the velocity of light c = 2.99792458×10^8 m/s

- Energy
  - Energy measured in **electronVolt,** 1eV=1.6022 x 10^{-19}J and higher products used (e.g. keV, MeV, GeV, TeV)
  - The momentum is $c\,p = \frac{v}{c}E$ and the kinetic energy $E_{kin} = E - E_0$. For ultra-relativistic particles $c\,p \approx E \approx E_{kin}$ and all three quantities referred as energy.
  - Momentum measured in eV/c
  - Heavy ions' energy quoted in terms of kinetic energy per nucleon
- Beam Current
  - Measured in Amperes
  - Intensity
  - Peak current
- Beam Power
- Magnetic field (induction)

A crash course on relativity

Relativity: 1905 - 2005

Einstein
International Year
of PHYSICS
2005
www.einstein2005.org.au

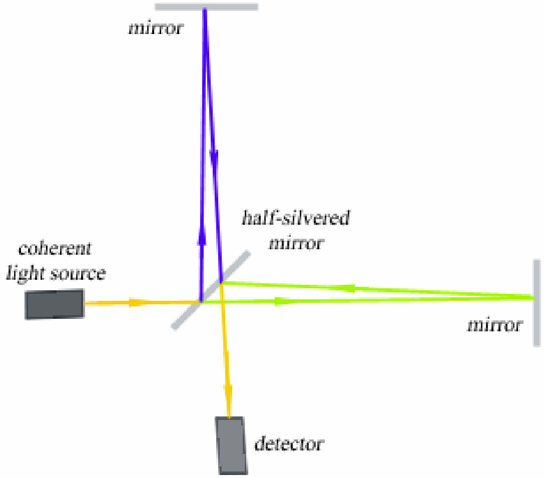
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - A few doubtful hypotheses and inconsistencies
    - A medium called "luminiferous ether" exists for the transport of electromagnetic waves
    - "Ether" has a small interaction with matter and is carried along with astronomical objects
    - Light propagates with speed $c = 3\times10^8$ m/s in "ether" but not invariant in all frames
    - Maxwell's equations are not invariant under Galilean transformations
    - In order to make electromagnetism compatible with classical mechanics, assume that light has speed c only in frames where source is at rest

Contradictions

- Star light aberration: a small shift in apparent positions of distant stars due to the finite speed of light and
- Fizeau and Foucault's experiments (1850) on the velocity of light in air and liquids
- Michelson-Morley experiment (1887) to detect motion of the earth through ether
- Lorentz-FitzGerland contraction hypothesis (1894 – 1904): perhaps bodies get compressed in the direction of their motion by a factor

Fizeau and Foucault

Michelson



$$\gamma = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$$

Postulates of Special Relativity

1. First Postulate (**Principle of Relativity**)
   Every **physical law is invariant** under inertial co-ordinate transformations. Thus, if an object in space-time obeys the mathematical equations describing a physical law in one inertial frame of reference, it must necessarily obey the same equations when using any other inertial frame of reference.
2. Second Postulate (**Invariance of the speed of light**)
   There exists an **absolute constant** $0 < c < \infty$ with the following property. If *A*, *B* are two events which have co-ordinates (x_1, x_2, x_3, t) (y_1, y_2, y_3, s) in one inertial frame *F*, and have co-ordinates (x'_1, x'_2, x'_3, t') and (y'_1, y'_2, y'_3, s') in another inertial frame *F*', then

$$\sqrt{(x'_1 - y'_1)^2 + (x'_2 - y'_2)^2 + (x'_3 - y'_3)^2} = c(s' - t')$$

if and only if

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = c(s - t)$$

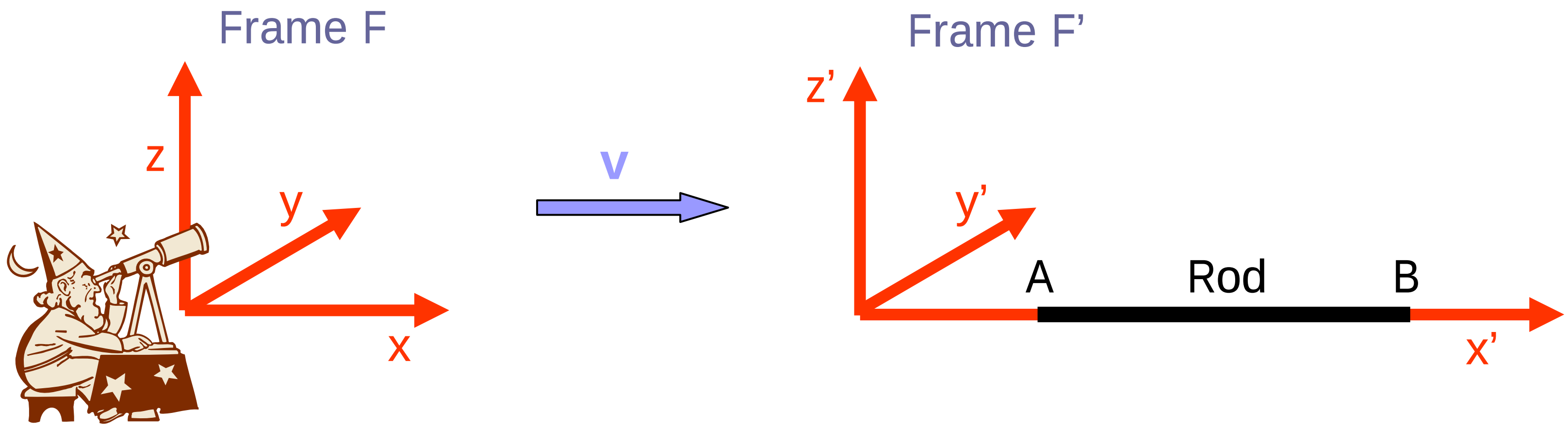
The Lorentz Transformation

- From a frame F (t,x,y,z) to a frame F'(t',x',y',z') moving with velocity v along the x-axis the space-time coordinates are transformed as:

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \text{ with } \begin{cases} \beta = \dfrac{v}{c} \\ \gamma = \dfrac{1}{\sqrt{(1-\beta^2)}} \end{cases}$$

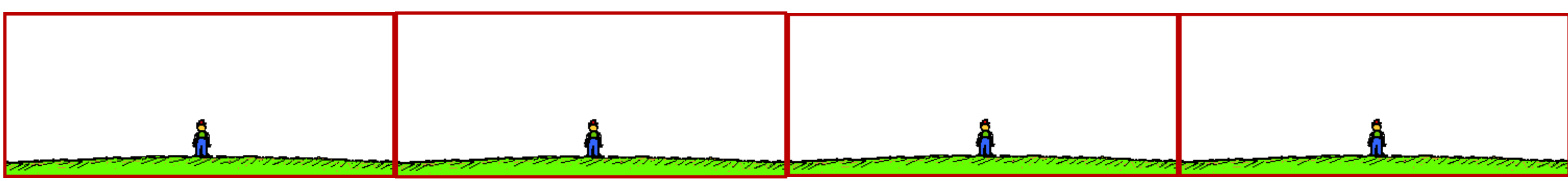
- The space-time interval is invariant under Lorentz transformations

$$ds = dx^2 + dy^2 + dz^2 - c\,t^2$$



Rod AB of length L' fixed in F' at x'_A, x'_B. Its length **L seen by the observer is contracted**

$$L' = x'_B - x'_A = \gamma(x_B - x_A) = \gamma L > L$$

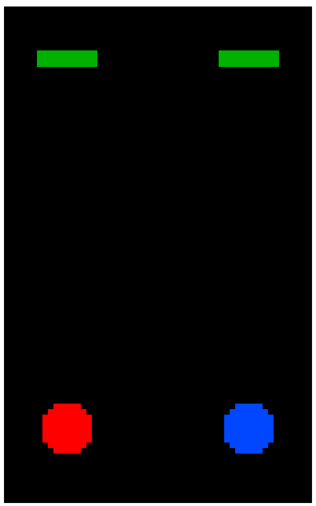


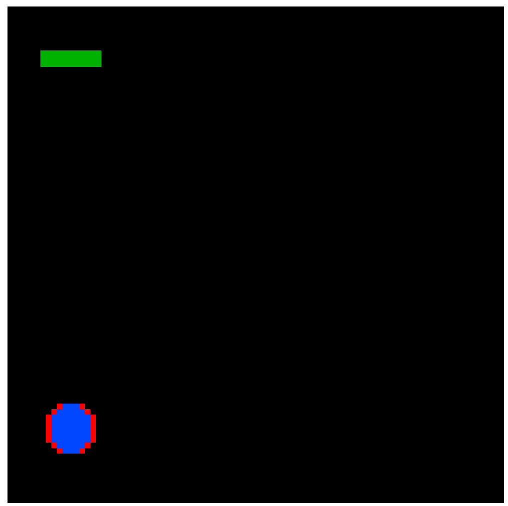
$v = 0.1\ c$ $\quad$ $v = 0.865\ c$ $\quad$ $v = 0.99\ c$ $\quad$ $v = 0.9999\ c$

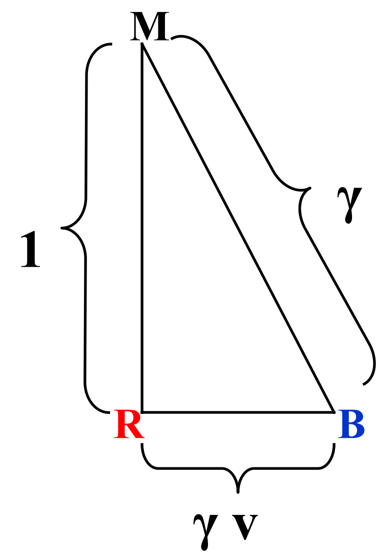
Time dilatation

Clock in frame F at point with coordinates (x,y,z) at different times t_A and t_B. In moving frame F' **the time difference Δt' is dilated**

$$\Delta t' = t'_B - t'_A = \gamma(t_B - t_A) = \gamma \Delta t > \Delta t$$







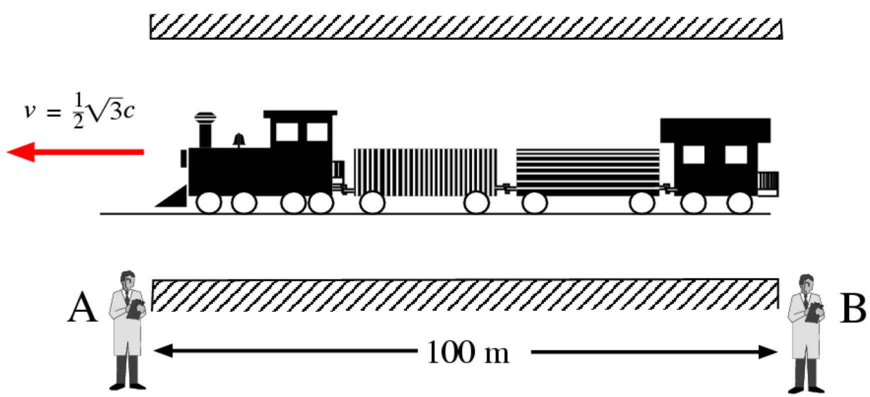
Red and **Blue** with identical clocks, (light beam bouncing off mirror). When **Red** and **Blue** at rest, tick and tocks are simultaneous

Blue moves with his mirror at velocity v. **Blue** measures the same time between ticks and tocks but according to **Red**, the clock of **Blue** runs slow

For $c = 1$, distance between mirror and **Red** RM=1 tick. **Red** thinks that distance between **Blue** and mirror is BM=γ ticks between **Blue** and himself RB=γ v. So,

$$\gamma^2 = (\gamma v)^2 + 1$$

Example: High Speed Train



- All clocks synchronised. F the frame of tunnel (and A,B) and F'=frame of train (with driver D and guard G)

$$x_A = 0 \quad x_B = 100$$
$$x'_D = 0 \quad x'_G = 100$$

- Observers A and B at entrance and exit of tunnel see train contracted

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 100 \times \left(1 - \frac{3}{4}\right)^{1/2} = 50\text{m}$$

- Tunnel moving relative to the driver and guard on train and they see train of100 m in length but tunnel of 50 m!

- What does B's clock read when guard goes into tunnel?

Coincident events (t_B, x_B) and (t'_G, x'_G)

$$x'_G = \gamma(x_B + vt_B) \quad \Rightarrow \quad 100 = 2(100 + vt_B)$$
$$\Rightarrow \quad t_B = -\frac{100}{c\sqrt{3}}$$

- What does guard's clock read as he enters?

$$x_B = \gamma(x'_G - vt'_G) \quad \Rightarrow \quad 100 = 2(100 - vt'_G)$$
$$\Rightarrow \quad t'_G = +\frac{100}{\sqrt{3}c}$$

- Where is the guard when his clock reads 0?

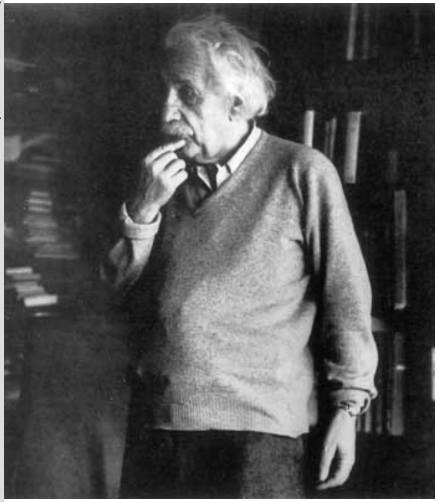
Set $t'_G = 0$ in $x = \gamma(x'_G - vt'_G)$

$$\Rightarrow \quad x = 2x'_G = 200 \ m$$

- So the guard is still 100m from tunnel entrance!!!

4-Vectors and Einstein's relations

- First note that $\frac{dt}{d\tau} = \gamma$ and $m = \gamma m_0$
- 4-Position: $\mathbf{X} = (c\, t, \mathbf{x})$
- 4-Velocity: $\mathbf{V} = \frac{d\mathbf{X}}{d\tau} = \gamma\, \frac{d\mathbf{X}}{dt} = \gamma\, \frac{d}{dt}(c\, t, \mathbf{x}) = \gamma\, (c, \mathbf{v})$
- 4-Acceleration: $\mathbf{A} = \frac{d\mathbf{V}}{d\tau} = \gamma\, \frac{d\mathbf{V}}{dt} = \gamma\, \frac{d}{dt}(c\, t, \mathbf{v}) = \gamma\, (\dot{\gamma} c, (\dot{\gamma \mathbf{v}}))$
- 4-Momentum: $\mathbf{P} = m_0 \mathbf{V} = m_0 \gamma\, (c, \mathbf{v}) = (m\, c, m\mathbf{v}) = (m\, c, \mathbf{p})$
- 4-Force: $\mathbf{F} = m_0 \mathbf{A} = \gamma(m_0 \dot{\gamma} c, m_0 (\dot{\gamma \mathbf{v}})) = \gamma(\dot{m} c, \mathbf{f})$
- Invariants: $\mathbf{V} \cdot \mathbf{V} = c^2$ and $\mathbf{P} \cdot \mathbf{P} = m_0^2 c^2$



Differentiating the momentum invariant $\mathbf{V} \cdot \frac{d\mathbf{P}}{d\tau} = 0$ and $\mathbf{V} \cdot \mathbf{F} = 0$

i.e. which gives $\dot{m} c^2 - \mathbf{v} \cdot \mathbf{f} = 0$. But the rate of change of kinetic energy

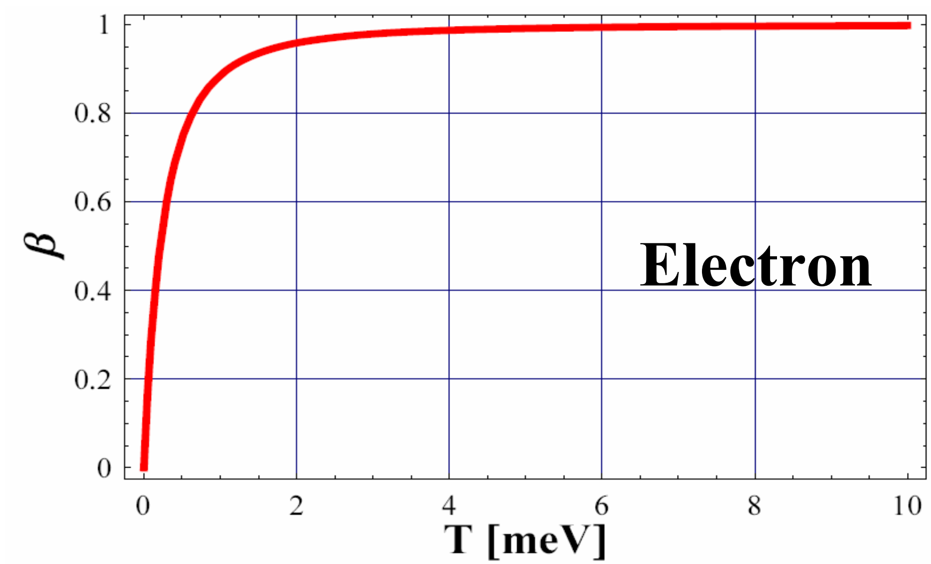
is $\frac{dT}{dt} = \mathbf{v} \cdot \mathbf{f} = \dot{m} c^2$ and by integrating we get $T = mc^2 + \text{constant}$.

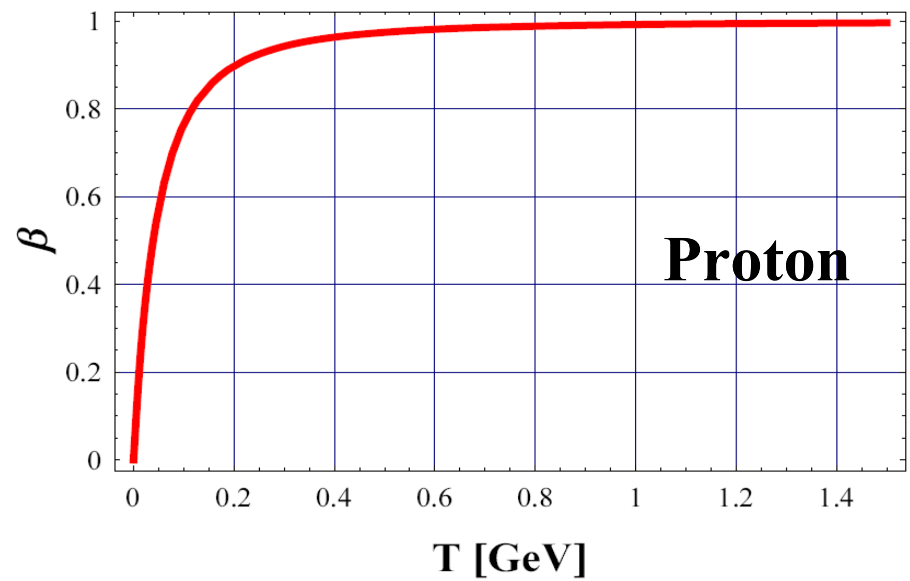
At rest $T = 0$, and then the constant is $-m_0 c^2$. Finally, we have

$$E \equiv T + E_0 = T + m_0 c^2 = mc^2$$

Velocity and Kinetic Energy

- Relative velocity $\beta = \frac{v}{c}$
- Relative velocity and Lorentz factor $\beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$
- Momentum $p = mv = \gamma m_0 v = \beta\gamma m_0 c$
- Kinetic energy $T = (m - m_0)c^2 = m_0 c^2(\gamma - 1)$
- For $v << c$ $\gamma \approx 1 + \frac{v^2}{2c^2}$ and $T = \frac{1}{2} m_0 v^2$





- Heavy particle become relativistic at higher energies
- When relativistic, small velocity change give big change in energy

Relationships between small parameter variations

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$		

Momentum conservation laws

- Equivalent expression for 4-momentum

$$\mathbf{P} = m_0 \mathbf{V} = m_0 \gamma \, (c, \mathbf{v}) = (m\, c, \mathbf{p}) = (\frac{E}{c}, \mathbf{p})$$

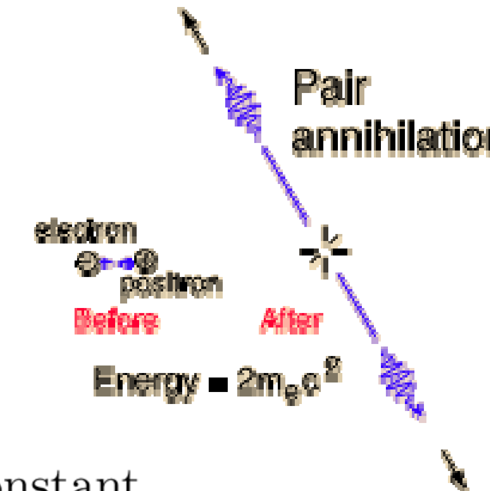
- From the momentum invariance we get

$$\frac{E^2}{c^2} = m_0^2 c^2 + \mathbf{p}^2$$

- As the 4-momentum is conserved

$$\sum_{i,\text{particles}} \mathbf{P_i} = \text{constant}$$

  - Total energy is conserved $\sum_{i,\text{particles}} E_i = \text{constant}$
  - Classical momentum is conserved $\sum_{i,\text{particles}} \mathbf{p_i} = \text{constant}$
  - Norm of the 4-momentum is conserved $\sum_{i,\text{particles}} |\mathbf{P_i}| = \text{constant}$



- Momentum is conserved but mass is not (mass is a form of energy)!!!

Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Mass Frame (CMF): Velocities are equal and opposite, total energy is E_{cm}.

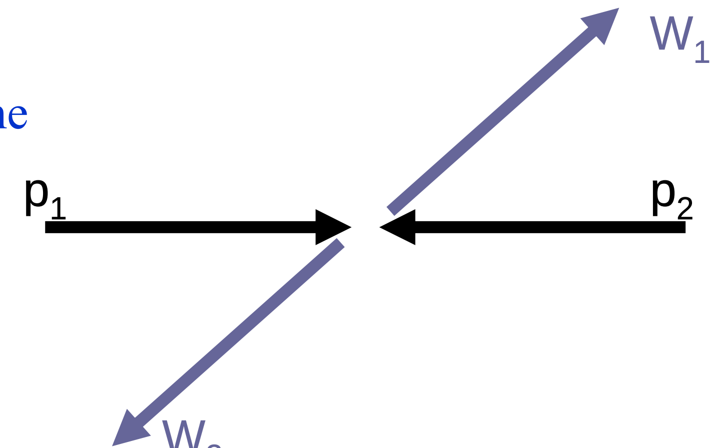
$$\mathbf{P_1} = (E_{\mathrm{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\mathrm{cm}}/(2c), -\mathbf{p})$$

- The quantity $(\mathbf{P_1} + \mathbf{P_2})^2$ is invariant.
- In the **CMF**, we have $(\mathbf{P_1} + \mathbf{P_2})^2 = E_{\mathrm{cm}}^2/c^2$.
- In general $(\mathbf{P_1} + \mathbf{P_2})^2 = \mathbf{P_1}^2 + \mathbf{P_2}^2 + 2\mathbf{P_1}\cdot\mathbf{P_2} = 2m_0^2c^2 + 2\mathbf{P_1}\cdot\mathbf{P_2}$
- In the **LF**, we have $\mathbf{P_1}\cdot\mathbf{P_2} = E_1 m_0$ $\qquad (\mathbf{P_1} + \mathbf{P_2})^2 = 2m_0 E$.
- And finally $E_{\mathrm{cm}}^2 = 2m_0c^2E$

More particle collisions

A proton p_1 collides with an anti-proton p_2 (same rest mass m_0), producing two particles W_1 and W_2 with mass M_0

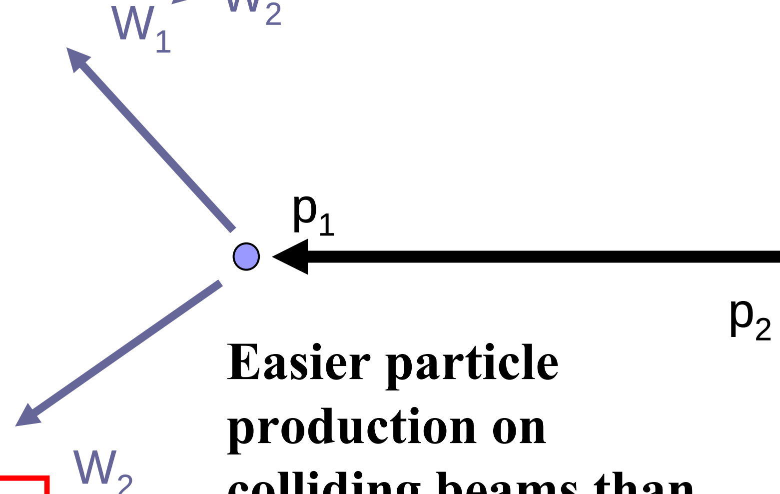
1. p_1 , p_2 with equal and opposite velocities in lab frame



$$(\mathbf{P_1} + \mathbf{P_2})^2 = E_{\rm cm}^2/c^2 = E_{\rm W}^2/c^2$$ and

$$E_{\rm cm} = E_{\rm W} \geq 2M_W c^2 \approx \boxed{180 \text{ GeV}}$$

2. p_1 with 0 velocity in lab frame.



$$E_{\rm cm}^2 = 2m_p c^2 E_{\rm lab}$$

$$E_{\rm lab} > \frac{2M_W^2 c^2}{m_p}$$ and as $$E = E_{\rm lab} - m_p c^2$$

we have $$E \geq (\frac{2M_W^2 c^2}{m_p} - m_p)c^2 \approx \boxed{1.3610^4 \text{ GeV}}$$

Easier particle production on colliding beams than fixed target experiments

Concepts of electromagnetism

Maxwell's Equations

- Accelerator physics: description of charged particle dynamics in the presence of electromagnetic fields
- Maxwell's equations relate Electric and Magnetic fields generated by charge and current distributions.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss law: divergence of the electric field gives the density of the sources.

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Faraday's law of induction: induced electric field in coil equal to negative rate of change of magnetic field

$$\nabla \cdot \mathbf{B} = 0$$

Gauss law for magnetism: there are no magnetic monopoles

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$$

Ampere-Maxwell's law: integral of magnetic field in closed loop proportional to current flowing in the loop (static electric field)

E = electric field	[V/m]	μ_0 (permeability of free space) = $4\pi\ 10^{-7}$ [C $V^{-1}m^{-1}$]
B = magnetic flux density	[T]	ε_0 (permittivity of free space) = 8.854 10^{-12} [V s $A^{-1}m^{-1}$]
ρ= charge density	[C/m^3]	c (speed of light) = 2.99792458 10^8 m/s
j = current density	[A/m^2]	$c^{-2} = \varepsilon_0\ \mu_0$

Vector and scalar potential

- Maxwell's equation is a set of coupled first order differential equations relating different components of E/M field
- Introduce potentials to reduce number of equations and unknowns
- From Gauss law of magnetism, we obtain the **magnetic vector potential**

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- From Faraday's law, we obtain an **electric scalar potential**

$$\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \quad \rightarrow \quad \exists \Phi : \quad \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \cdot \Phi$$

- Inserting these equations back to Gauss and Ampere's law

$$\nabla^2 \Phi + \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\epsilon_0}$$ and $$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

Considering the **Lorentz gauge invariants**

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \Lambda$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

and choose $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

to get the decoupled equations

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}$$

Lorentz force

- Force on charged particles moving in an electromagnetic field:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- In accelerator physics: electric fields used for particle acceleration and magnetic fields for particle guidance (but not exclusively)
  - Integrate Lorentz force with the path length to get kinetic energy

$$\Delta T = \int \mathbf{F} d\mathbf{s} = q \int \mathbf{E} d\mathbf{s} + \cancel{q \int (\mathbf{v} \times \mathbf{B}) \mathbf{v} dt}$$

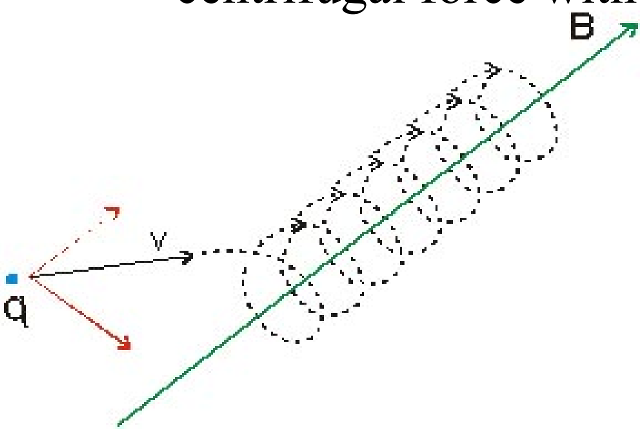
  - Kinetic energy is changed from the presence of electric but not magnetic field
- Relativistic equation of motion

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m_0 \gamma \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Notions of particle beam dynamics

- The particle acceleration is $\dot{\mathbf{p}} = \dot{(m_0\gamma\mathbf{v})} = m_0\gamma\dot{\mathbf{v}} + m_0\mathbf{v}\dot{\gamma}$ with $\dot{\gamma} = \gamma^3\dot{v}\beta/c$, giving $\dot{\mathbf{p}} = m_0(\gamma\dot{\mathbf{v}} + \gamma^3\beta\mathbf{v}\dot{v}/c)$
- For a force parallel to particle propagation $\dot{\mathbf{p}}_{||} = m_0\gamma^3\dot{\mathbf{v}}_{||}$
- For a force perpendicular to propagation $\dot{\mathbf{p}}_{\perp} = m_0\gamma\dot{\mathbf{v}}_{\perp}$
- Parallel acceleration is much more efficient!!!

- Example: Motion in a uniform, constant magnetic field
  - Spiraling along the magnetic field with constant energy. Equate centrifugal force with Lorentz force to get $\frac{m_0\gamma v^2}{\rho} = q\,v\,B$



from where the radius is $\rho = \frac{m_0\gamma v}{qB} = |\frac{p}{qB}|$

and the frequency $\omega = \frac{\nu}{\rho} = \frac{qB}{m_0\gamma} = \frac{qBc^2}{E}$

Electromagnetic waves

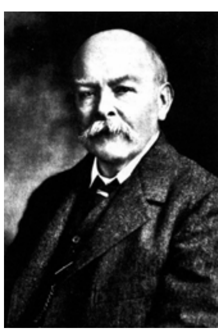
- Maxwell's equations predict the existence of electromagnet waves, later discovered by Hertz.

- Assume no charges and no currents:
  - Take the curl of Faraday's law and replace curl of magnetic field by using Ampere's law

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-\frac{\partial}{\partial t}\mathbf{B}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

  - Use the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$ and Gauss law to obtain a 3D wave equation

$$\boxed{\nabla^2\mathbf{E} = \frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} + \frac{\partial^2\mathbf{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}}$$



- E/M waves carry energy, with a flow (power per unit area) described by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

Nature of electromagnetic waves

- Plane wave with angular frequency ω travelling in the direction of the wave vector **k**

$$\mathbf{E} = \mathbf{E_0} \exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})]$$

$$\mathbf{B} = \mathbf{B_0} \exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})]$$

- From Gauss' laws

$$\mathbf{k}\cdot\mathbf{E} = \mathbf{k}\cdot\mathbf{B} = 0$$

i.e. fields are transverse to each other and wave propagation

- From Faraday's law

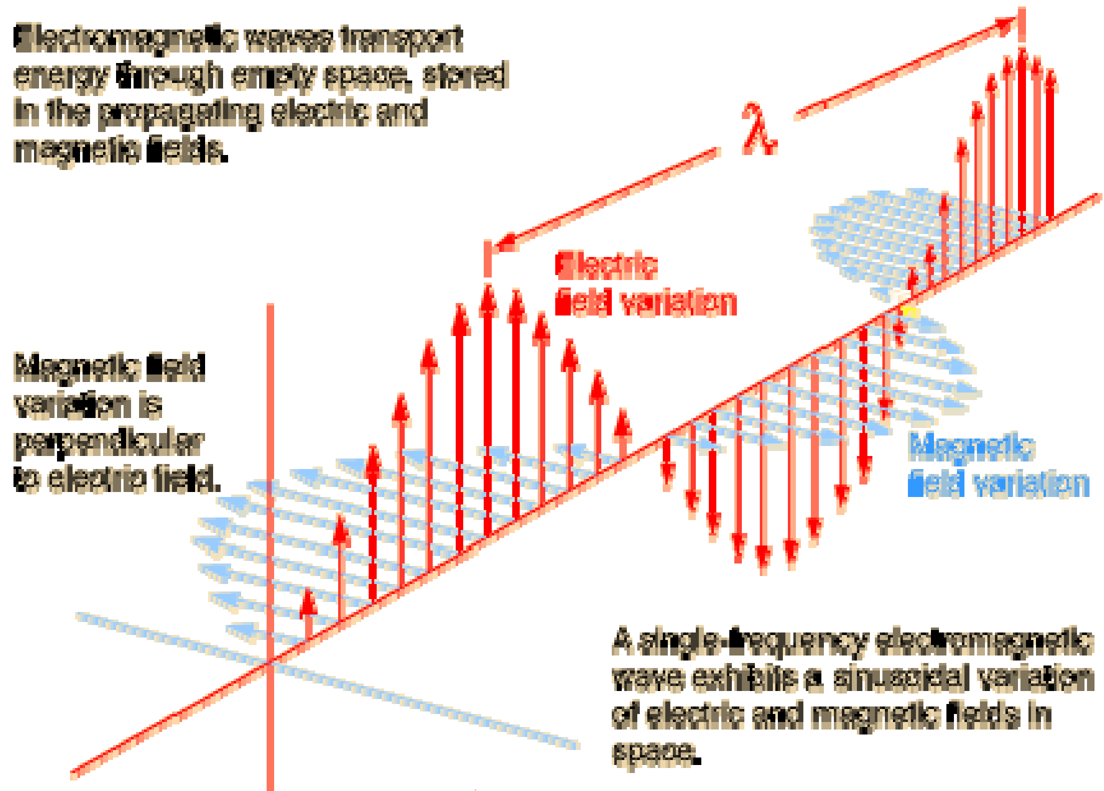
$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}$$

- From Ampere-Maxwell's law

$$\mathbf{k}\times\mathbf{B} = -\omega/c^2\mathbf{E}$$

- We have that the velocity of propagation in vacuum is

$$\frac{|\mathbf{E}|}{|\mathbf{B}|} = \frac{\omega}{|\mathbf{k}|} = c$$



with wavelength $\lambda = \frac{2\pi}{|\mathbf{k}|}$ and frequency $\nu = \frac{\omega}{2\pi}$