Betatron functions

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20th June – 1st July 2005

Outline

- General solutions of Hill's equations
  - Floquet theory
- Betatron functions
- Transfer matrices revisited
  - General and periodic cell
- General transport of betatron functions
  - Drift
  - Beam waist
- Normalized coordinates

Solution of Betatron equations

- Symmetries of beam transport

$$x'' + K_x(s)\ x = 0$$
$$y'' + K_y(s)\ y = 0$$

- Betatron equations are linear with periodic coefficients $K_x(s) = K_x(s+C)\ ,\ \ K_y(s) = K_y(s+C)$
- **Floquet theorem** states that the solutions are

$$u(s) = Aw(s)\cos(\psi(s) + \psi_0)$$

where **w(s), ψ(s)** are periodic with the same period

$$w(s) = w(s+C)\ ,\ \ \psi(s) = \psi(s+C)$$

- Note that solutions resemble the one of harmonic oscillator

$$u(s) = A\cos(\psi(s) + \psi_0)$$

- Substitute solution in Betatron equations

$$u'' + K(s)\ u = A\underbrace{(2w'\psi' + w\psi'')}_{0}\sin(\psi + \psi_0) + A\underbrace{(w'' - w\psi'^2 + Kw)}_{0}\cos(\psi + \psi_0) = 0$$

Betatron functions

- By multiplying with w the coefficient of **sin**

$$2w'w\psi' + w^2\psi'' = (w^2\psi')' = 0$$

- Integrate to get $\psi = \int \frac{ds}{w^2(s)}$

- Replace **ψ'** in the coefficient of **cos** and obtain

$$w^3(w'' + K_x w) = 1$$

- Define the **Betatron** or **twiss** or **lattice functions** (Courant-Snyder parameters)

$$\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \\ \gamma(s) &\equiv \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}$$

Betatron motion

- The on-momentum linear betatron motion of a particle is described by

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \psi_0)$$

with $\alpha, \ \beta, \ \gamma$ the twiss functions $\alpha(s) = -\frac{\beta(s)'}{2}, \quad \gamma = \frac{1+\alpha(s)^2}{\beta(s)}$

ψ the **betatron phase** $\psi(s) = \int \frac{ds}{\beta(s)}$

and the **beta function** β is defined by the **envelope equation**

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$$

- By differentiation, we have that the **angle** is

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s) + \psi_0) + \alpha(s)\cos(\psi(s) + \psi_0)\right)$$

Courant-Snyder invariant

- Eliminating the angles by the position and slope we define the **Courant-Snyder invariant**

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$

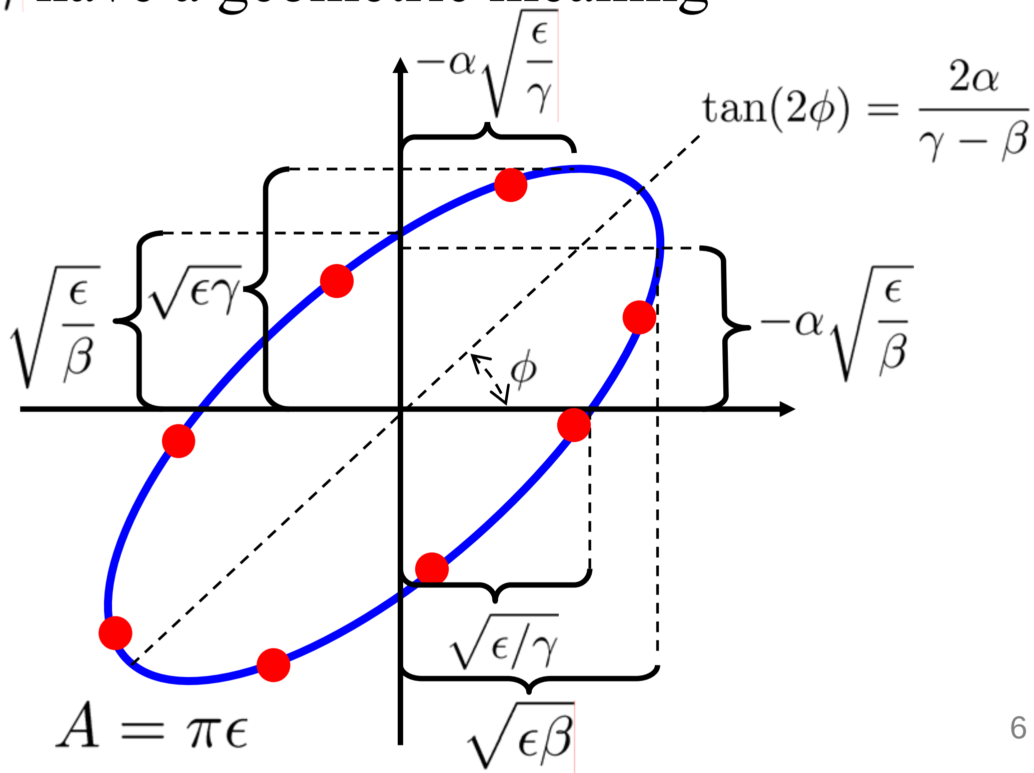
- This is an ellipse in phase space with area **πε**
- The twiss functions $\alpha,\ \beta,\ \gamma$ have a geometric meaning

- The beam envelope is

$$E(s) = \sqrt{\epsilon\beta(s)}$$

- The beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$



General transfer matrix

- From equation for position and angle we have

$$\cos(\psi(s)+\psi_0) = \frac{u}{\sqrt{\epsilon\beta(s)}}\ , \quad \sin(\psi(s)+\psi_0) = \sqrt{\frac{\beta(s)}{\epsilon}}u' + \frac{\alpha(s)}{\sqrt{\epsilon\beta(s)}}u$$

- Expand the trigonometric formulas and set ψ(0)=0 to get the transfer matrix from location 0 to s

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathcal{M}_{0\to s} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

with

$$\mathcal{M}_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta(s)\beta_0}\sin\Delta\psi \\ \frac{(a_0 - a(s))\cos\Delta\psi - (1+\alpha_0\alpha(s))\sin\Delta\psi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\Delta\psi - \alpha_0\sin\Delta\psi) \end{pmatrix}$$

and $\Delta\psi = \int_0^s \frac{ds}{\beta(s)}$ the **phase advance**

Periodic transfer matrix

- Consider a periodic cell of length **C**
- The optics functions are $\beta_0 = \beta(C) = \beta\ , \quad \alpha_0 = \alpha(C) = \alpha$

  and the phase advance $$\mu = \int_0^C \frac{ds}{\beta(s)}$$

- The transfer matrix is

$$\mathcal{M}_C = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- The cell matrix can be also written as

$$\mathcal{M}_C = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the **Twiss matrix** $\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

Stability conditions

- From the periodic transport matrix $\mathrm{Trace}(\mathcal{M}_C) = 2\cos\mu$ and the following stability criterion

$$|\mathrm{Trace}(\mathcal{M}_C)| < 2$$

- In a ring, the **tune** is defined from the 1-turn phase advance

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

i.e. number betatron oscillations per turn

- From a general transfer matrix for a cell $\mathcal{M}_C = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ we get

$$\cos\mu = \frac{1}{2}(m_{11}+m_{22}),\ \beta = \frac{m_{12}}{\sin\mu},\ \alpha = \frac{m_{11}-m_{22}}{2\sin\mu},\ \gamma = -\frac{m_{21}}{\sin\mu}$$

Transport of Betatron functions

- For a general matrix between position 1 and 2

$$\mathcal{M}_{s_1 \to s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{and the inverse} \quad \mathcal{M}_{s_2 \to s_1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

- Equating the invariant at the two locations

$$\epsilon = \gamma_{s_2} {u_{s_2}}^2 + 2\alpha_{s_2} u_{s_2} u'_{s_2} + \beta_{s_2} u'^2_{s_2} = \gamma_{s_1} {u_{s_1}}^2 + 2\alpha_{s_1} u_{s_1} u'_{s_1} + \beta_{s_1} u'^2_{s_1}$$

and eliminating the transverse positions and angles

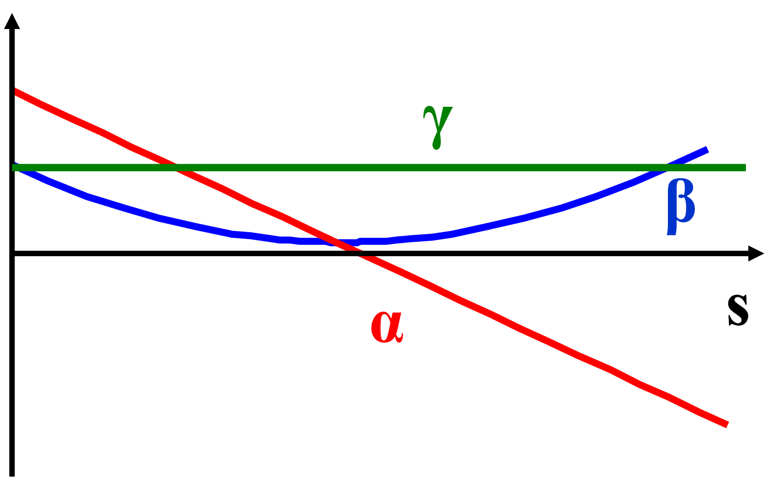
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{22}m_{12} \\ m_{21}^2 & 2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

Example I: Drift

- Consider a drift with length **s**
- The transfer matrix is $\mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$
- The betatron transport matrix is $\begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$

from which

$$\begin{aligned} \beta(s) &= \beta_0 - 2s\alpha_0 + s^2\gamma_0 \\ \alpha(s) &= \alpha_0 - s\gamma_0 \\ \gamma(s) &= \gamma_0 \end{aligned}$$



Simplified method for betatron transport

- Consider the beta matrix $\mathcal{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ the matrix

$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and its transpose $\mathcal{M}^T_{1\to 2} = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$

- It can be shown that

$$\mathcal{B}_2 = \mathcal{M}_{1\to 2} \cdot \mathcal{B}_1 \cdot \mathcal{M}^T_{1\to 2}$$

- Application in the case of the drift

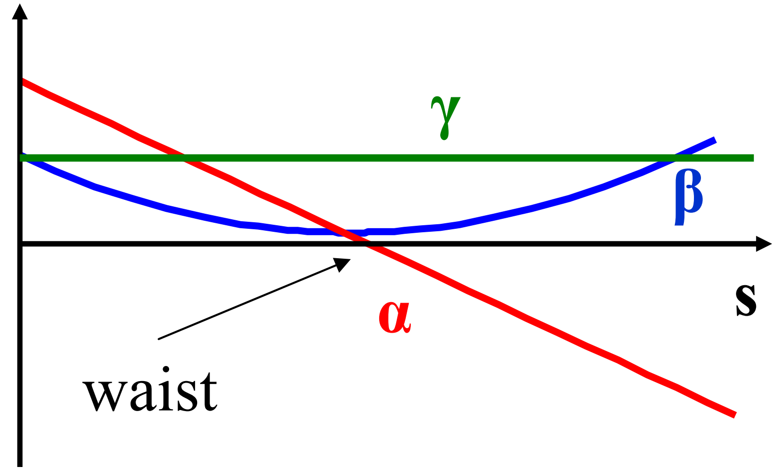
$$\mathcal{B} = \mathcal{M}_{\text{drift}} \cdot \mathcal{B}_0 \cdot \mathcal{M}^T_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} \beta_0 - 2s\alpha_0 + s^2\gamma_0 & -\alpha_0 + s\gamma_0 \\ -\alpha_0 + s\gamma_0 & \gamma_0 \end{pmatrix}$$

Example II: Beam waist

- For beam waist **α=0** and occurs at **s = α_0/γ_0**
- Beta function grows quadratically and is minimum in waist



$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

- The beta at the waste for having beta minimum $\frac{d\beta(s)}{d\beta_0} = 0$

at the entrance and exit of a drift with length **L** is $\beta_0 = \frac{s}{2}$

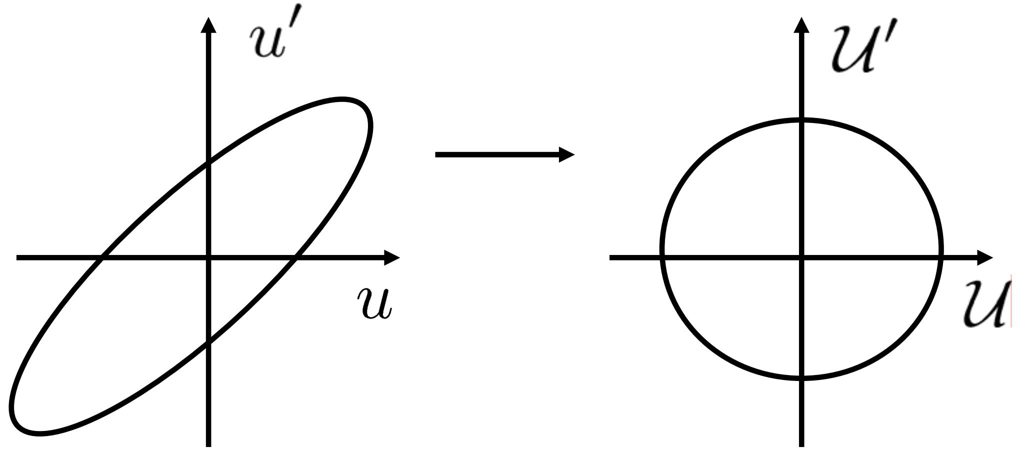
- The phase advance of a drift is $\mu = \int_0^{L/2} \frac{ds}{\beta(s)} = \arctan(\frac{L}{2\beta_0})$

which is **π/2** when $\beta_0 \to \infty$. Thus, for a drift $\mu = \leq \pi$

Normalized coordinates

- Introduce new variables $\mathcal{U} = \frac{u}{\sqrt{\beta}}\ , \ \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ , \ \phi = \frac{\psi}{\nu}$
- In matrix form

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}$$



- **Floquet transformation** transforms phase space in circles
- System becomes harmonic oscillator with frequency ***ν***

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{\epsilon} \begin{pmatrix} \cos(\nu\phi) \\ -\sin(\nu\phi) \end{pmatrix}$$