Electron dynamics

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Outline

- Radiation damping
  - Synchrotron oscillations
  - Betatron oscillations
  - Robinson theorem
- Radiation integrals
- Equilibrium emittances

Damping of synchrotron oscillations

- Consider the differential equation of the energy for longitudinal motion $\Delta\ddot{E} + \alpha_s \Delta\dot{E} + \Omega\Delta E = 0$ with damping coefficient $\alpha_s = \frac{1}{2T_0}\frac{dU}{dE}$
  where U is the energy requirement per turn of the particle, and T_0 the revolution period
- A particle with momentum spread follows a dispersive trajectory with dispersion D
  $$ds' = (1 + \frac{\Delta x}{\rho})ds = (1 + \frac{D}{\rho}\frac{\Delta E}{E})ds$$
- The energy requirement per turn can be obtained by the integral of the radiated power in one revolution
  $$U = \oint P_s dt = \oint P_s ds'/c = \frac{1}{c}\oint P_s(1 + \frac{D}{\rho}\frac{\Delta E}{E})ds$$
- Differentiating with respect to the energy
  $$\frac{dU}{dE} = \frac{1}{c}\oint \left[\frac{dP_s}{dE} + \frac{D}{\rho}(\frac{dP_s}{dE}\frac{\Delta E}{E} + \frac{P_s}{E})\right] ds$$

Damping of synchrotron oscillations II

- Taking into account that the average energy spread around the ring should be zero the previous integral is written:

$$\frac{dU}{dE} = \frac{1}{c} \oint \left(\frac{dP_s}{dE} + \frac{D}{\rho} \frac{P_s}{E} \right) ds$$

- Setting $\mathcal{C} = \dfrac{e^4 c^3}{6\pi\varepsilon_0 (m_0 c^2)^4}$ and taking into account the definition of the magnetic rigidity, the expression of the radiation power is written $P_s = \mathcal{C} E^2 B^2$
- Its derivative with respect to the energy gives

$$\frac{dP_s}{dE} = 2\frac{P_s}{E}(1 + Dk\rho)$$

where we used the identity $\dfrac{dB}{dE} = \dfrac{dB}{dx}\dfrac{dx}{dE} = \dfrac{dB}{dx}\dfrac{D}{E} = Bk\rho\dfrac{D}{E}$

- Replacing in the integral at the top,

$$\frac{dU}{dE} = \frac{2U_0}{E} + \frac{1}{cE} \oint DP_s \left(2k\rho + \frac{1}{\rho} \right) ds$$

Damping of synchrotron oscillations III

- Replacing in the last integral the expression of the power

$$\oint DP_s\left(2k\rho+\frac{1}{\rho}\right)ds=\frac{\mathcal{C}E^4}{e^2c^2}\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds$$

and taking into account that $U_0=\frac{1}{c}\oint P_s ds=\frac{\mathcal{C}E^4}{e^2c^3}\oint\frac{ds}{\rho^2}$

the damping of synchrotron motion is written

$$\alpha_s=\frac{1}{2T_0}\frac{dU}{dE}=\frac{U_0}{2ET_0}(2+\mathcal{D})$$

with the damping partition number defined as

$$\mathcal{D}=\frac{\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds}{\oint\frac{ds}{\rho^2}}$$

- Entirely defined by the lattice!
- Bending magnets and quads are usually separated and the damping partition number is usually extremely small

Damping of vertical oscillations

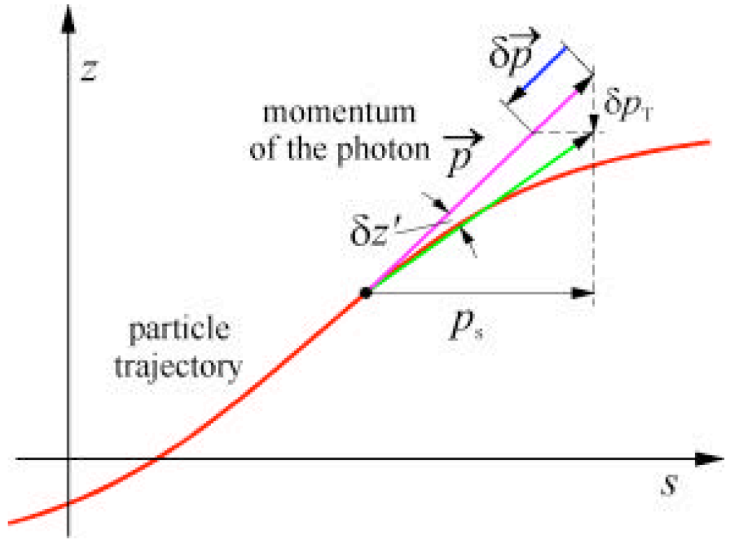
- Consider solution of Hill's equations in the vertical plane, assuming that the alpha function is zero

$$y = A\cos\phi \;\text{ and }\; y' = -\frac{A}{\beta(s)}\sin\phi$$

- The Betatron oscillation amplitude is

$$A^2 = y^2 + [\beta(s)y']^2$$

- Synchrotron radiation emitted in the direction of motion of electron, whose momentum changes $\mathbf{p}^* = \mathbf{p} - \delta\mathbf{p}$



- Longitudinal component of the momentum restored by RF cavity, i.e. angle is further reduced $y' = \frac{\delta p_\perp}{|\mathbf{p}|}$
- The change in energy will not affect the vertical position but the angle changes proportionally $\delta y' = y'\frac{\delta E}{E}$

Damping of vertical oscillations II

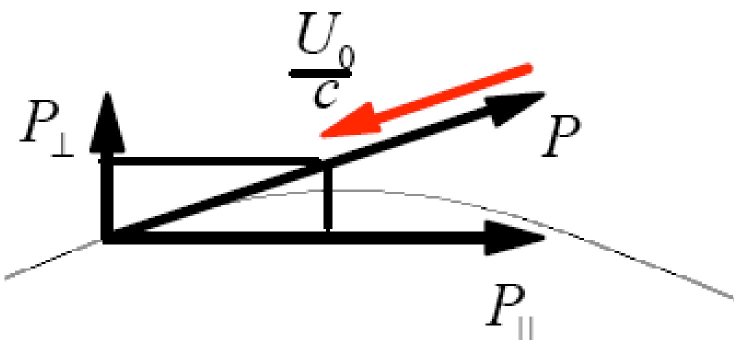
- The change of the amplitude becomes

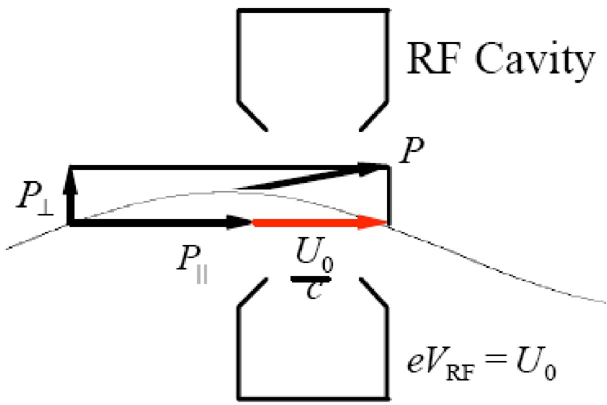
$$\delta(A^2) = \beta(s)\delta(y'^2) \Rightarrow A\delta A = -\beta^2(s)y'^2\frac{\delta E}{E}$$

- By averaging over all angles $\langle \beta^2(s)y'^2 \rangle = \frac{A^2}{2\pi}\int_0^{2\pi} \sin^2\phi d\phi = \frac{A^2}{2}$ and $A\langle\delta A\rangle = -\frac{A^2}{2}\frac{\delta E}{E}$
- Summing up the energy losses for a full turn $\frac{\Delta A}{A} = -\frac{U_0}{2E}$
- Thus, in one turn the amplitudes are damped with a constant

$$\alpha_y = -\frac{\Delta A}{A\Delta t} = \frac{U_0}{2ET_0}$$

- Note that the key for Betatron damping is the energy recovery by the RF cavities





Damping of horizontal oscillations

- The horizontal motion is described by

$$x = x_\beta + x_e = A\cos\phi + D\frac{\Delta E}{E} \quad \text{and} \quad x' = x'_\beta + x'_e = -\frac{A}{\beta(s)}\sin\phi + D'\frac{\Delta E}{E}$$

- Energy change *u* due to photon emission results in a change of the dispersive part but not of the total coordinates so that

$$\delta x_\beta = -\delta x_e = D\frac{u}{E} \quad \text{and} \quad x\delta x'_\beta = -\delta x'_e = D'\frac{u}{E}$$

- The change of the Betatron amplitude $A^2 = x_\beta^2 + \left[\beta(s)x'_\beta\right]^2$ becomes $A\delta A = -(Dx_\beta + \beta^2(s)D'x'_\beta)\frac{u}{E}$

- The energy loss in an element *dl* is written

$$u = -\frac{P_s(x_\beta)}{c}dl = -\frac{1}{c}\left(P_s + 2\frac{P_s}{B}\frac{dB}{dx}x_\beta\right)\left(1 + \frac{x_\beta}{\rho}\right)ds$$

- Substituting to the change in amplitude and averaging over the angles (and some patience…)

$$\frac{\Delta A}{A} = -(1-\mathcal{D})\frac{U_0}{2E}$$ and the damping coefficient $$\alpha_x = \frac{U_0}{2ET_0}(1-\mathcal{D})$$

Robinson theorem

- Grouping the damping constants and introducing the three damping times and damping partition numbers

$$\alpha_s = \frac{1}{\tau_s} = \frac{U_0}{2ET_0}(2+\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_s$$

$$\alpha_y = \frac{1}{\tau_y} = \frac{U_0}{2ET_0} = \frac{U_0}{2ET_0}\mathcal{J}_y$$

$$\alpha_x = \frac{1}{\tau_x} = \frac{U_0}{2ET_0}(1-\mathcal{D}) = \frac{U_0}{2ET_0}\mathcal{J}_x$$

- The **Robinson** theorem (1958) states that the sum of the damping partition number is an invariant

$$\mathcal{J}_x + \mathcal{J}_y + \mathcal{J}_s = 4$$

- In storage ring with separated function magnets, $\mathcal{D} \ll 1$ and

$$\mathcal{J}_x = 1\ , \quad \mathcal{J}_y = 1\ , \quad \mathcal{J}_s = 2$$

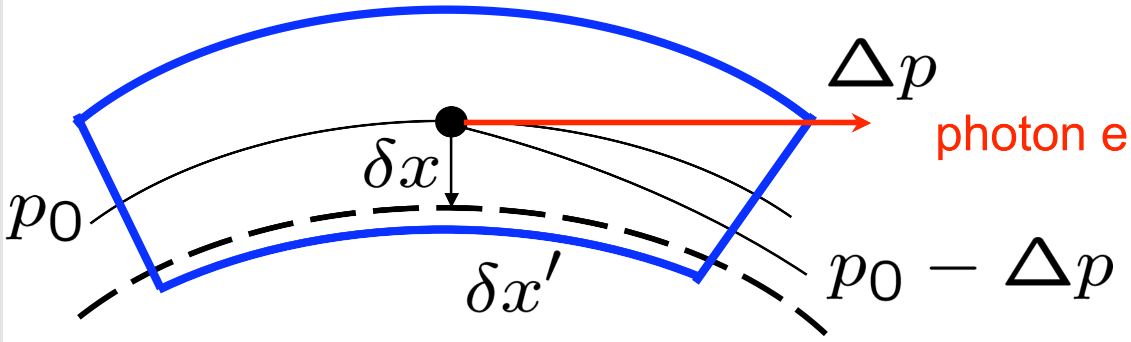
Transverse emittances

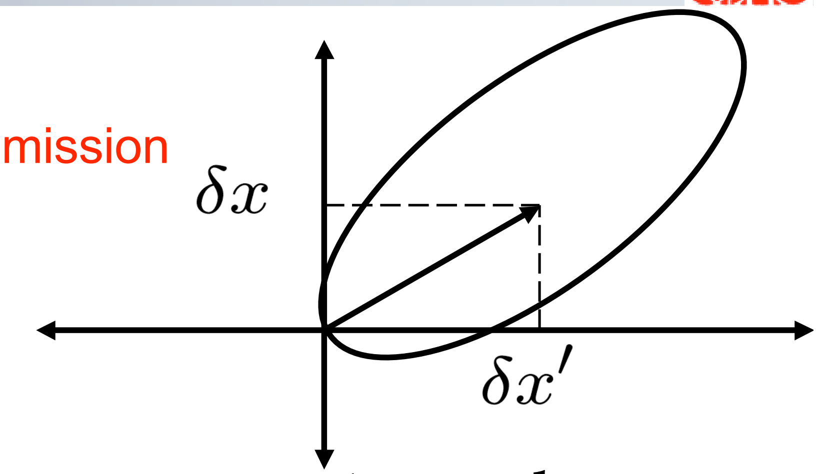
- Recall that the brilliance is proportional to the flux and inversely proportional to the emittance

$$\mathrm{Brilliance} = \frac{\mathrm{Flux}}{4\pi^2 \epsilon_x \epsilon_y}$$

- It is necessary to reduce the emittance as much as possible
- If the only effect was radiation damping, the transverse emittances would be damped to zero.
- BUT, photons are emitted in energy bursts in localized areas and horizontal Betatron oscillations are excited as well (quantum fluctuations)
- Vertical emittance can become very small and only excited by coupling with the horizontal
- Electrons are influenced by this stochastic effect and eventually loose memory (unlike hadrons)

Dispersion emittance





- Assume electron along nominal momentum orbit with initially negligible emittance
- After photon emission with momentum Δp, electron's momentum becomes p_0-Δp and the trajectory becomes

$$\delta x = D\frac{\Delta p}{p} \quad \text{and} \quad \delta x' = D'\frac{\Delta p}{p}$$

- Now, the emittance is non-negligible becoming $\epsilon = \frac{\Delta p}{p}\mathcal{H}(s)$

with $\mathcal{H}(s) = \beta(s)D(s)'^2 + 2\alpha(s)D(s)D'(s) + \gamma(s)D(s)^2$

the "**dispersion**" **emittance**

Equilibrium emittance

- Averaging over all photon energies and emission probabilities, the **equilibrium emittance** is

$$\epsilon_x = \frac{C_q \gamma^2}{\mathcal{J}_x} \frac{\oint \frac{\mathcal{H}_x(s)}{|\rho_x|^3} ds}{\oint \frac{1}{\rho_x^2} ds} \text{ with } C_q = \frac{55}{32\sqrt{3}} \frac{h}{m_0 c} = 3.83 \times 10^{-13} \text{ m}$$

- For isomagnetic ring with separated function magnets the equilibrium emittance is written

$$\epsilon_x = 1.47 \times 10^{-6} \frac{E^2}{\rho} \frac{1}{l_{\text{bend}}} \int_0^{l_{\text{bend}}} \mathcal{H}_x(s) ds$$

- The integral depends on the optics functions on the bends
- It gets small for small horizontal beta and dispersion, but this necessitates strong quadrupoles
- Smaller bending angle and lower energy reduce emittance

Radiation integrals

$$\mathcal{I}_1 = \oint \frac{D}{\rho} ds$$ Momentum compaction factor $$\alpha_c = \frac{\mathcal{I}_1}{2\pi R}$$

$$\mathcal{I}_2 = \oint \frac{1}{\rho^2} ds$$ Energy loss per turn $$U_0 = \frac{C_\gamma}{2\pi} E^4 \mathcal{I}_2$$

$$\mathcal{I}_3 = \oint \frac{1}{|\rho|^3} ds$$ Equilibrium energy spread $$\sigma_\delta^2 = C_q \gamma^2 \frac{\mathcal{I}_3}{2\mathcal{I}_2 + \mathcal{I}_4}$$

$$\mathcal{I}_4 = \oint \frac{D}{\rho^3}(1 + 2k\rho^2) ds$$

$$\mathcal{J}_x = 1 - \frac{\mathcal{I}_4}{\mathcal{I}_2} \ , \quad \mathcal{J}_s = 2 + \frac{\mathcal{I}_4}{\mathcal{I}_2} \ , \quad \mathcal{D} = \frac{\mathcal{I}_4}{\mathcal{I}_2}$$

Damping partition numbers

$$\mathcal{I}_5 = \oint \frac{\mathcal{H}}{|\rho^3|} ds$$ Equilibrium betatron emittance $$\epsilon_x = C_q \gamma^2 \frac{\mathcal{I}_5}{\mathcal{I}_2 - \mathcal{I}_4}$$

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