Linear imperfections and correction

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- Steering error and closed orbit distortion
- Gradient error and beta beating correction
- Linear coupling and correction
- Chromaticity

Closed orbit distortion

- Causes
  - Dipole field errors
  - Dipole misalignments
  - Quadrupole misalignments
    - Consider the displacement of a particle **δx** from the ideal orbit . The vertical field is

$$B_y = G\bar{x} = G(x+\delta x) = \underbrace{Gx}_{\text{quadrupole}} + \underbrace{G\delta x}_{\text{dipole}}$$

- Remark: Dispersion creates a closed orbit distortion for off-momentum particles $\delta x = D(s)\frac{\delta p}{p}$
- Effect of orbit errors in any multi-pole magnet

$$B_y = b_n \bar{x}^n = b_n (x+\delta x)^n = b_n (\underbrace{x^n}_{\textbf{2(n+1)-pole}} + \underbrace{n\delta x x^{n-1}}_{\textbf{2n-pole}} + \underbrace{\frac{n(n-1)}{2}(\delta x)^2 x^{n-2}}_{\textbf{2(n-1)-pole}} + \cdots + \underbrace{(\delta x)^n}_{\textbf{dipole}})$$

- **Feed-down**

Effect of single dipole kick

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}} \ , \quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}} u + \sqrt{\beta} u' \ , \quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu} \int \frac{ds}{\beta(s)}$$

- The Hill's equations are written $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2 \mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator $\mathcal{U} = \mathcal{U}_0 \cos(\nu\phi)$
- Consider a single dipole kick $\delta u'(\pi) = \frac{\delta(Bl)}{B\rho}$ at **$\varphi=\pi$**
- Then

$$\mathcal{U}'(\pi) = -\mathcal{U}_0 \nu \sin(\pi\nu) = \frac{d\mathcal{U}}{d\phi}\Big|_{\phi=\pi} = \frac{d\mathcal{U}}{ds}\frac{ds}{d\phi}\Big|_{s=k} = \frac{d\mathcal{U}}{ds}\Big|_{s=k} \beta(k)\nu = \sqrt{\beta(k)}\frac{du}{ds}\Big|_{s=k}$$

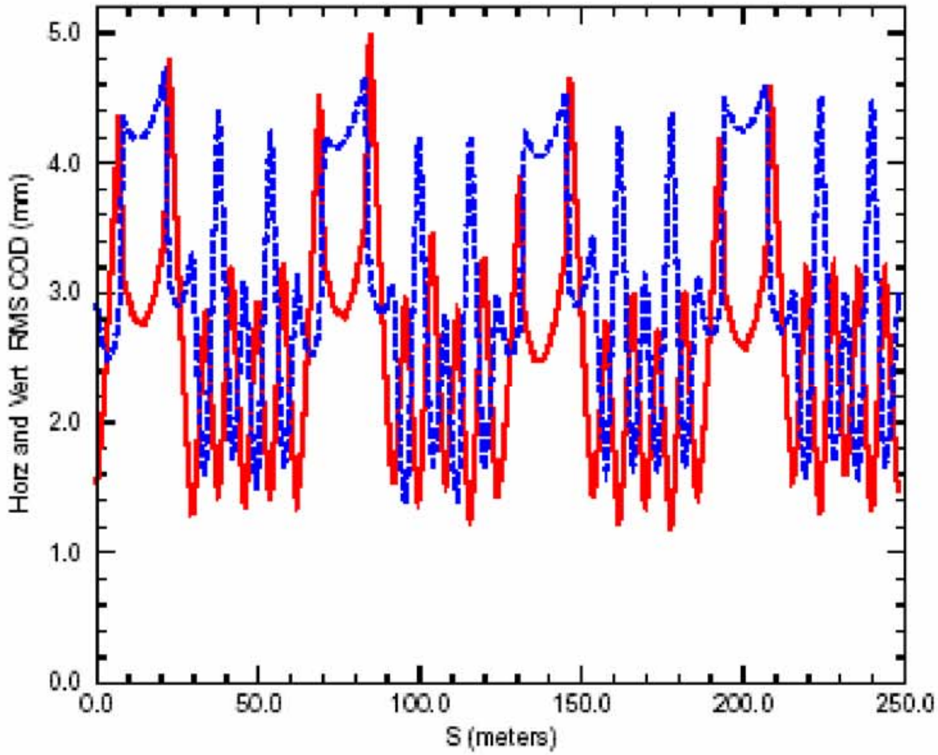
and $\mathcal{U}_0 = \frac{\sqrt{\beta(k)}}{2|\sin(\nu\pi)|} \delta u'(\pi)$ with $\frac{\delta u'(\pi)}{2} = \frac{du}{ds}\Big|_{s=k} = \frac{\delta(Bl)}{2B\rho}$

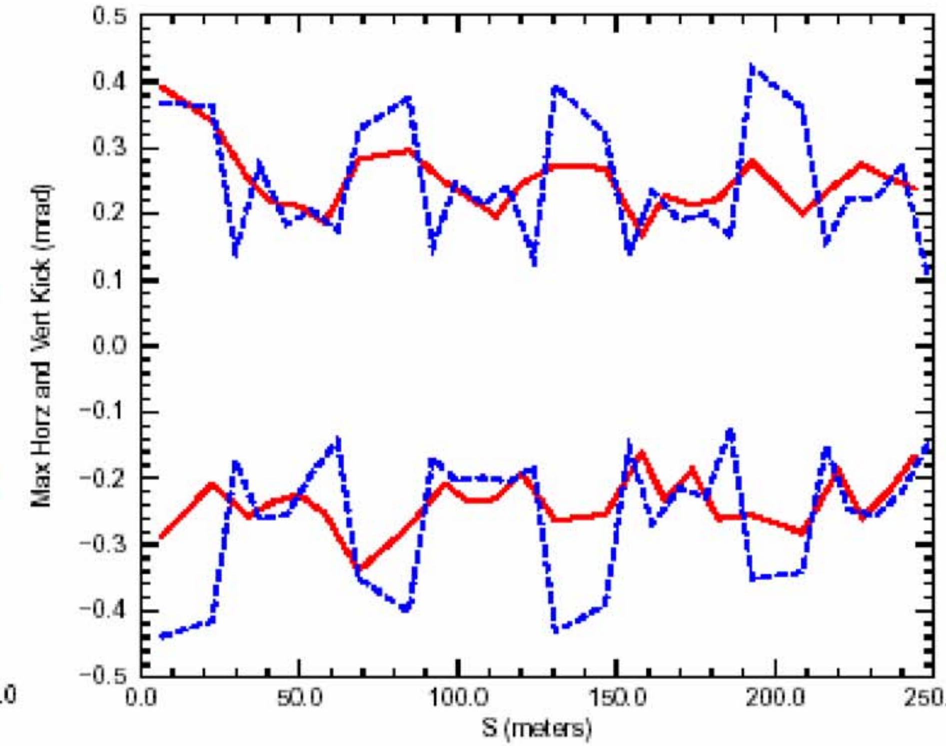
and in the old coordinates

$$u(s) = \sqrt{\beta(s)} \mathcal{U}_0 \cos(\nu\phi(s)) = \underbrace{\frac{\sqrt{\beta(s)\beta(k)}}{2\sin(\pi\nu)}}_{} \frac{\delta(Bl)}{B\rho} \cos(\nu\phi(s))$$

Maximum distortion amplitude

Example: Orbit distortion for the SNS ring





- In the SNS accumulator ring, the beta function is **6m** in the dipoles and **30m** in the quadrupoles.
- Consider dipole error of **δy'=1mrad**
- The tune is 6.2
- The maximum orbit distortion in the dipoles is $y_0 = \frac{\sqrt{6 \cdot 6}}{2\sin(6.2\pi)} \cdot 10^{-3} \approx 5\text{mm}$
- For quadrupole displacement with 0.5mrad error the distortion is $y_0 \approx 2.5\text{cm}$!!!

Closed orbit distortion

Horizontal-vertical orbit distortion (Courant and Snyder 1957)

$$\delta_{x,y}(s) = -\frac{\sqrt{\beta_{x,y}}}{2\sin(\pi Q_{x,y})}\int_s^{s+C}\frac{\Delta B(\tau)}{B\rho}\sqrt{\beta_{x,y}}\cos(|\pi Q_{x,y}+\psi_{x,y}(s)-\psi_{x,y}(\tau)|)d\tau$$

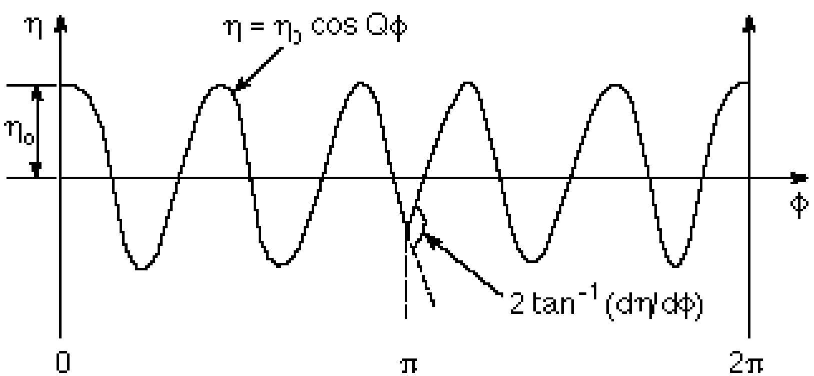
with $\Delta B(\tau)$ the equivalent magnetic field error at $s=\tau$.

Approximate errors as delta functions in n locations:

$$\delta_{x,y;i} = -\frac{\sqrt{\beta_{x,y;i}}}{2\sin(\pi Q_{x,y})}\sum_{j=i+1}^{i+n}\phi_{x,y;j}\sqrt{\beta_{x,y;j}}\cos(|\pi Q_{x,y}+\psi_{x,y;i}-\psi_{x,y;j}|)$$

with $\phi_{x,y;j}$ kick produced by jth element:

- $\phi_j = \frac{\Delta B_j L_j}{B\rho} \rightarrow$ dipole field error
- $\phi_j = \frac{B_j L_j \sin\theta_j}{B\rho} \rightarrow$ dipole roll
- $\phi_j = \frac{G_j L_j \Delta x,y_j}{B\rho} \rightarrow$ quadrupole displacement



Many orbit errors' effect

- Consider random distribution of errors in N magnets
- The expectation value is given by

$$< u(s) >= \frac{\sqrt{\beta(s)}}{2\sqrt{(2)}\sin(\pi\nu)} \sum_i \beta_i \delta u'_i = \frac{\sqrt{\beta(s)\langle\beta\rangle}\sqrt{N}}{2\sqrt{(2)}\sin(\pi\nu)} \frac{(\delta Bl)_{\mathrm{rms}}}{B\rho}$$

- Example:
  - In the SNS ring, there are **32** dipoles and **54** quadrupoles
  - The expectation value of the orbit distortion in the dipoles

$$y_0 = \frac{\sqrt{6 \cdot 6}\sqrt{32}}{2\sqrt{2}\sin(6.2\pi)} \cdot 10^{-3} \approx 2\mathrm{cm}$$

  - And in the quadrupoles

$$y_0 = \frac{\sqrt{30 \cdot 30}\sqrt{54}}{2\sqrt{2}\sin(6.2\pi)} \cdot 10^{-3} \approx 13\mathrm{cm}$$

Transport of orbit distortion due to dipole kick

- Consider a transport matrix between positions 1 and 2

$$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

- The transport of transverse coordinates is written as

$$\begin{aligned} u_2 &= m_{11}u_1 + m_{12}u'_1 \\ u'_2 &= m_{21}u_1 + m_{22}u'_1 \end{aligned}$$

- Consider a single dipole kick at position 1 $\delta u'_1 = \frac{\delta(Bl)}{B\rho}$
- Then, the first equation may be rewritten

$$u_2 + \delta u_2 = m_{11}u_1 + m_{12}(u'_1 + \delta u'_1) \rightarrow \delta u_2 = m_{12}\delta u'_1$$

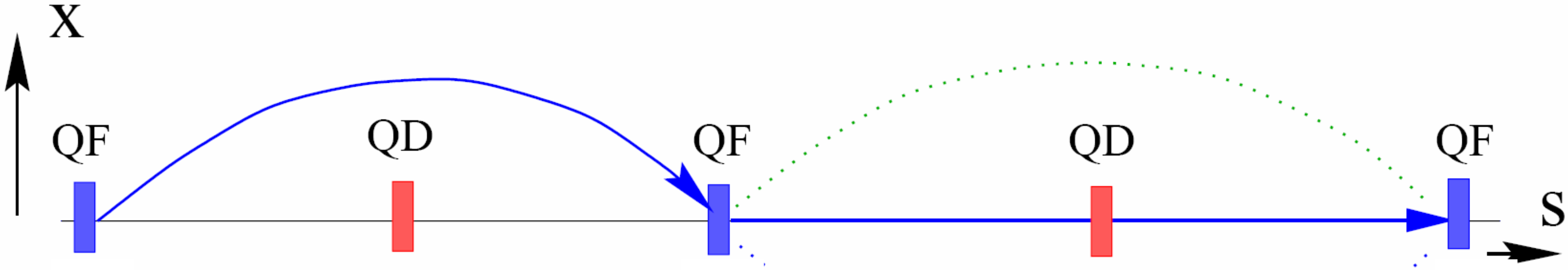
- Replacing the coefficient from the general betatron matrix

$$\delta u_2 = \sqrt{\beta_1\beta_2}\sin(\phi_{12})\delta u'_1$$

Closed orbit correction for the SNS ring

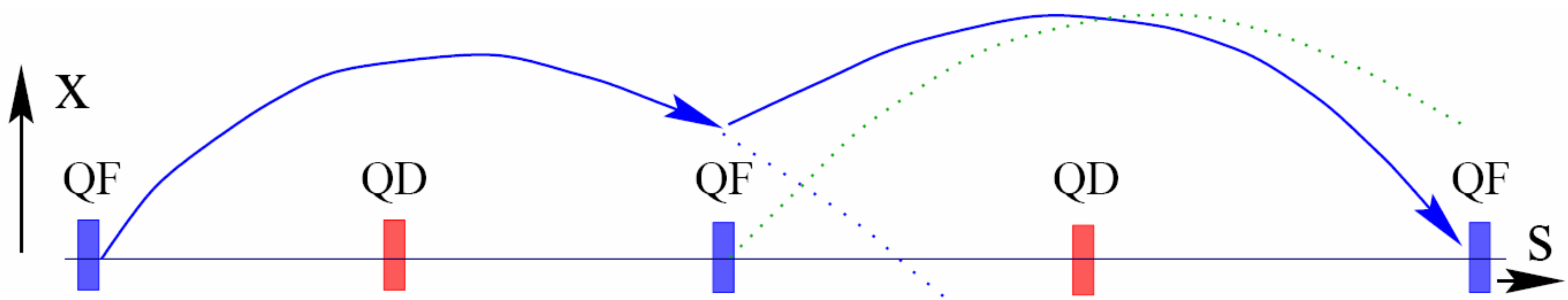
- Orbit distortion usually dominated by misalignments in the quadrupoles
- Place horizontal and vertical dipole correctors close to the corresponding quads
- Simulate (random distribution of errors) or measure orbit in Beam position monitors (downstream of the correctors)
- Minimize orbit distortion with several methods
  - Globally
    - Harmonic
    - Most efficient corrector
    - Least square
  - Locally
    - Bumps
    - Singular Value Decomposition (SVD)

Orbit bumps



- **2-bump**: Only good for phase advance equal π between correctors
- Sensitive to lattice and BPM errors
- Large number of correctors

$$\delta u_2' = \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \delta u_1'$$



- **3-bump**: works for any lattice
- Need large number of correctors
- No control of angles

$$\frac{\sqrt{\beta_1}}{\sin \phi_{23}} \delta u_1' = \frac{\sqrt{\beta_2}}{\sin \phi_{31}} \delta u_2' = \frac{\sqrt{\beta_3}}{\sin \phi_{12}} \delta u_3'$$

Gradient error and optics distortion

- Key issue for the performance -> super-periodicity preservation -> only structural resonances excited
- Broken super-periodicity -> excitations of all resonances
- Causes
  - Errors in quadrupole strengths (random and systematic)
  - Injection elements
  - Higher-order multi-pole magnets and errors
- Observables
  - Tune-shift
  - Beta-beating
  - Excitation of integer and half integer resonances

Gradient error

- Consider the transfer matrix for one turn

$$\mathcal{M}_0 = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ -\gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - \alpha_0 \sin(2\pi Q) \end{pmatrix}$$

- Consider a gradient error in a quad. In thin element approximation the quad matrix with and without error are

$$m_0 = \begin{pmatrix} 1 & 0 \\ -K_0(s)ds & 1 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 1 & 0 \\ -(K_0(s) + \delta K)ds & 1 \end{pmatrix}$$

- The new 1-turn matrix is $\mathcal{M} = m m_0^{-1} \mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -\delta K ds & 1 \end{pmatrix} \mathcal{M}_0$ which yields

$$\mathcal{M}_0 = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ \delta K ds(\cos(2\pi Q) - \alpha_0 \sin(2\pi Q)) - \gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - (\delta K ds \beta_0 + \alpha_0) \sin(2\pi Q) \end{pmatrix}$$

Gradient error and tune-shift

- Consider a new matrix after 1 turn with a new tune $\chi = 2\pi(Q+\delta Q)$

$$\mathcal{M}^\star = \begin{pmatrix} \cos(\chi)+\alpha_0\sin(\chi) & \beta_0\sin(\chi) \\ -\gamma_0\sin(\chi) & \cos(\chi)-\alpha_0\sin(\chi) \end{pmatrix}$$

- The traces of the two matrices describing the 1-turn should be equal $\quad \mathrm{Tra}(\mathcal{M}^\star) = \mathrm{Tra}(\mathcal{M})$

  which gives $2\cos(2\pi Q) - \delta K ds \beta_0 \sin(2\pi Q) = 2\cos(2\pi(Q+\delta Q))$
- Developing the left hand side

$$\cos(2\pi(Q+\delta Q)) = \cos(2\pi Q)\underbrace{\cos(2\pi\delta Q)}_{1} - \sin(2\pi Q)\underbrace{\sin(2\pi\delta Q)}_{2\pi\delta Q}$$

- and finally $\quad 4\pi\delta Q = \delta K ds \beta_0$
- For a quadrupole of finite length, we have

$$\delta Q = \frac{1}{4\pi}\int_{s_0}^{s_0+l} \delta K \beta_0 ds$$

Gradient error and beta distortion

- Consider the unperturbed transfer matrix for one turn

$$\mathcal{M}_0 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = B \cdot A \;\; \text{with} \;\; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\; \text{and} \;\; \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

- Introduce a gradient perturbation between the two matrices

$$\mathcal{M}_0^\star = \begin{pmatrix} m_{11}^\star & m_{12}^\star \\ m_{21}^\star & m_{22}^\star \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\delta K ds & 1 \end{pmatrix} A$$

- Recall that $m_{12} = \beta_0 \sin(2\pi Q)$ and write the perturbed term as

$$m_{12}^\star = (\beta_0 + \delta\beta)\sin(2\pi(Q + \delta Q)) = \delta\beta \sin(2\pi Q) + 2\pi\delta Q \beta_0 \cos(2\pi Q)$$

- On the other hand

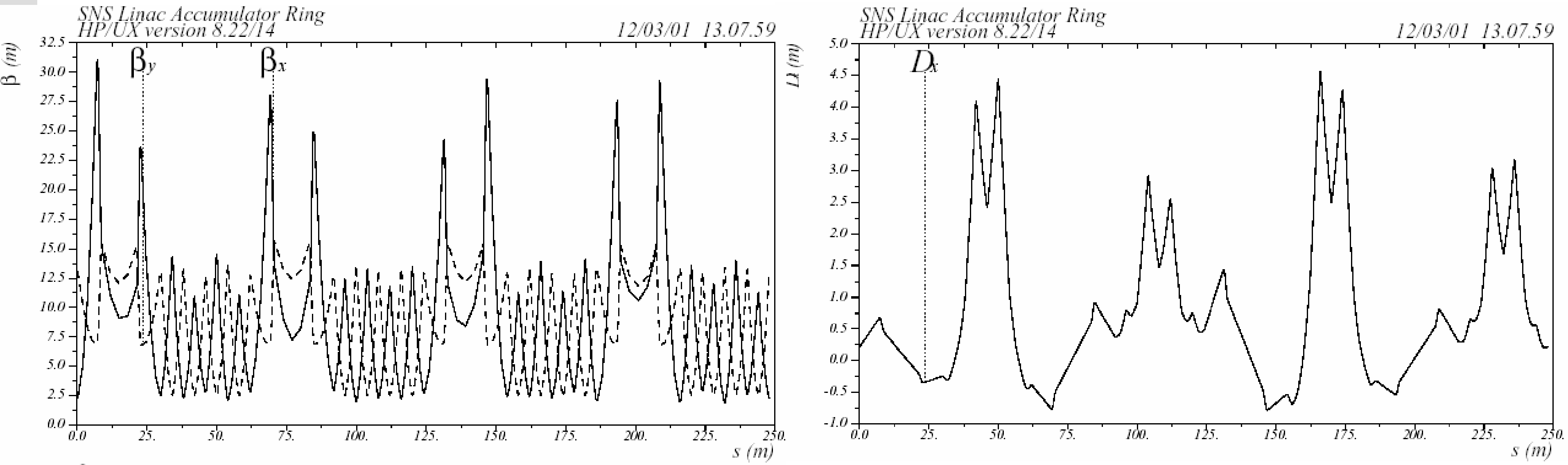
$$m_{12}^\star = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12}} - a_{12}b_{12}\delta K ds = \beta_0 \sin(2\pi Q) - a_{12}b_{12}\delta K ds$$

- and $a_{12} = \sqrt{\beta_0\beta(s_1)} \sin\psi, \;\; b_{12} = \sqrt{\beta_0\beta(s_1)} \sin(2\pi Q - \psi)$

- Equating the two terms and integrating through the quad

$$\frac{\delta\beta}{\beta_0} = -\frac{1}{2\sin(2\pi Q)} \int_{s_1}^{s_1+l} \beta(s)\delta K(s) \cos(2\psi - 2\pi Q) ds$$

Example: Gradient error in the SNS storage ring



- Consider **18** focusing arc quads in the SNS ring with **1%** gradient error. In this location **β=12m**. The length of the quads is **0.5m**
- The tune-shift is $\delta Q = \frac{1}{4\pi} 18 \cdot 12 \frac{0.01}{5.6567} 0.5 = 0.015$
- For a random distribution of errors the beta beating is

$$\frac{\delta\beta}{\beta_0}_{\rm rms} = -\frac{1}{2\sqrt{2}|\sin(2\pi Q)|}\left(\sum_i \delta k_i^2 \beta_i^2\right)^{1/2}$$

- Optics functions beating > 20% by putting random errors (1% of the gradient) in high dispersion quads of the SNS ring
- Justifies the choice of TRIM windings strength

Gradient error correction

- Windings on the core of the quadrupoles (TRIM)
- Simulation by introducing random distribution of quadrupole errors
- Compute the tune-shift and the optics function beta distortion
- Move working point close to integer and half integer resonance
- Minimize beta wave or quadrupole resonance width with TRIM windings
- To correct certain resonance harmonics N, strings should be powered accordingly
- Individual powering of TRIM windings can provide flexibility and beam based alignment of BPM

Linear coupling

- Betatron motion is coupled in the presence of skew quadrupoles
- The field is $(B_x, B_y) = k(x, y)$ and Hill's equations are coupled
- Motion still linear with two new eigen-mode tunes, which are always split. In the case of a thin quad:

$$\delta Q \propto |k|\sqrt{\beta_x \beta_y}$$

- Coupling coefficients

$$|C_\pm| = \left| \frac{1}{2\pi} \oint ds\, k(s) \sqrt{\beta_x(s)\beta_y(s)} e^{i(\phi_x \pm \phi_y - (Q_x \pm Q_y - q_\pm) 2\pi s / C)} \right|$$

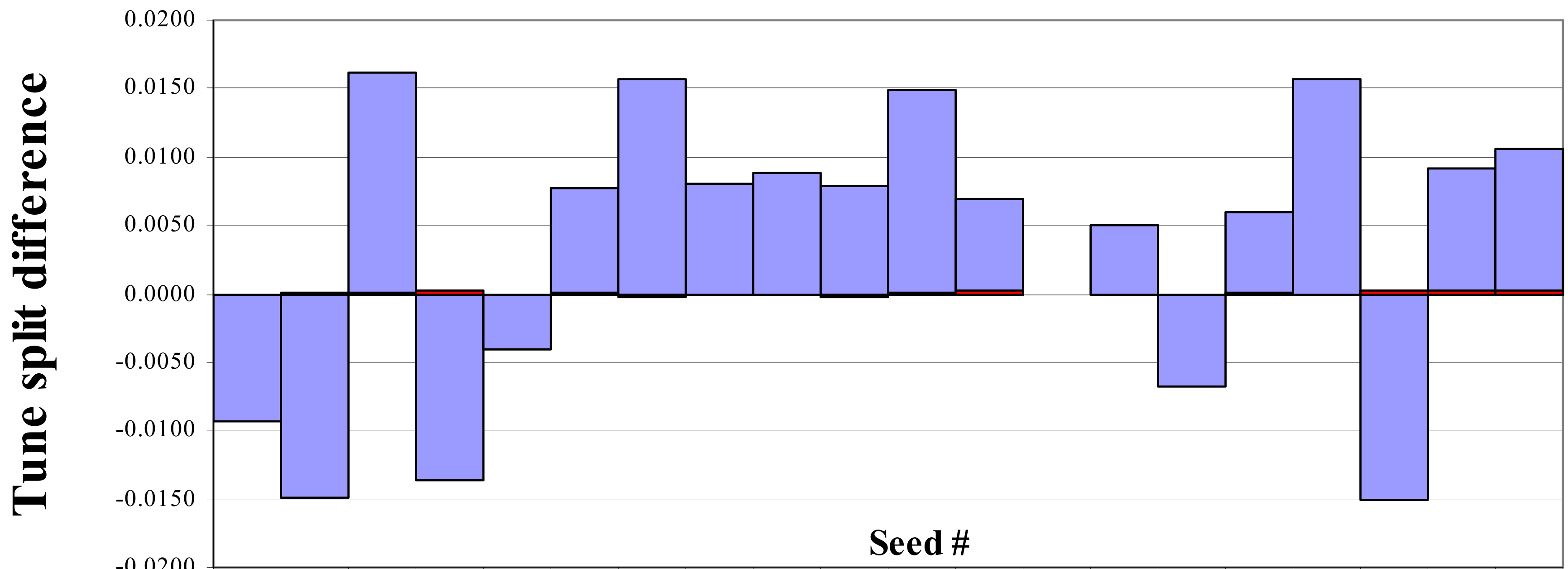
- As motion is coupled, vertical dispersion and optics function distortion appears
- Causes:
  - Random rolls in quadrupoles
  - Skew quadrupole errors
  - Off-sets in sextupoles

Linear coupling correction

- Introduce skew quadrupole correctors
- Simulation by introducing random distribution of quadrupole errors
- Correct globally/locally coupling coefficient (or resonance driving term)
- Correct optics distortion (especially vertical dispersion)
- Move working point close to coupling resonances and repeat
- Correction especially critical for flat beams

Example: Coupling correction for the SNS ring

- Local decoupling by super period using 16 skew quadrupole correctors
- Results of Q_x=6.23 Q_y=6.20 after a 2 mrad quad roll
- Additional 8 correctors used to compensate vertical dispersion



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Before Correction	-0.009	-0.014	0.016	-0.013	-0.004	0.007	0.015	0.008	0.008	0.007	0.014	0.006	0.000	0.005	-0.006	0.006	0.015	-0.015	0.009	0.010
After correction	0.000	0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	-0.000	0.000	-0.000	0.000	0.000	0.000

Chromaticity

- Linear equations of motion depend on the energy (term proportional to dispersion)
- Chromaticity is defined as: $\xi_{x,y} = -\frac{\delta Q_{x,y}}{\delta P/P}$
- Recall that the gradient is $K = \frac{G}{B\rho} = \frac{eG}{P} \rightarrow \frac{\delta K}{K} = \mp \frac{\delta P}{P}$
- This leads to dependence of tunes and optics function on energy
- For a linear lattice the tune shift is:

$$\delta Q_{x,y} = \frac{1}{4\pi} \oint \beta_{x,y} \delta K(s) ds = \frac{1}{4\pi} \frac{\delta P}{P} \oint \beta_{x,y} K(s) ds$$

- So the natural chromaticity is:

$$\xi_{x,y} = -\frac{1}{4\pi} \oint \beta_{x,y} K(s) ds$$

Example: Chromaticity in the SNS ring

- In the SNS ring, the natural chromaticity is –7.
- Consider that momentum spread $\frac{\delta P}{P} = \pm 1$
- The tune-shift for off-momentum particles is

$$\delta Q_{x,y} = \xi_{x,y} \frac{\delta P}{P} = \pm 0.07$$

- In order to correct chromaticity introduce particles which can focus off-momentum particle

↓

Sextupoles

Chromaticity from sextupoles

- The sextupole field component in the x-plane is: $B_y = \frac{S}{2}x^2$
- In an area with non-zero dispersion $x = x_0 + D\frac{\delta P}{P}$
- Than the field is

$$B_y = \frac{S}{2}x_0^2 + \underbrace{SD\frac{\delta P}{P}x_0}_{\text{quadrupole}} + \underbrace{\frac{S}{2}D^2\frac{\delta P}{P}^2}_{\text{dipole}}$$

- Sextupoles introduce an equivalent focusing correction

$$\delta K = SD\frac{\delta P}{P}$$

- The sextupole induced chromaticity is

$$\xi_{x,y}^{S} = -\frac{1}{4\pi}\oint \beta_{x,y}(s)S(s)D_x(s)ds$$

- The total chromaticity is the sum of the natural and sextupole induced chromaticity

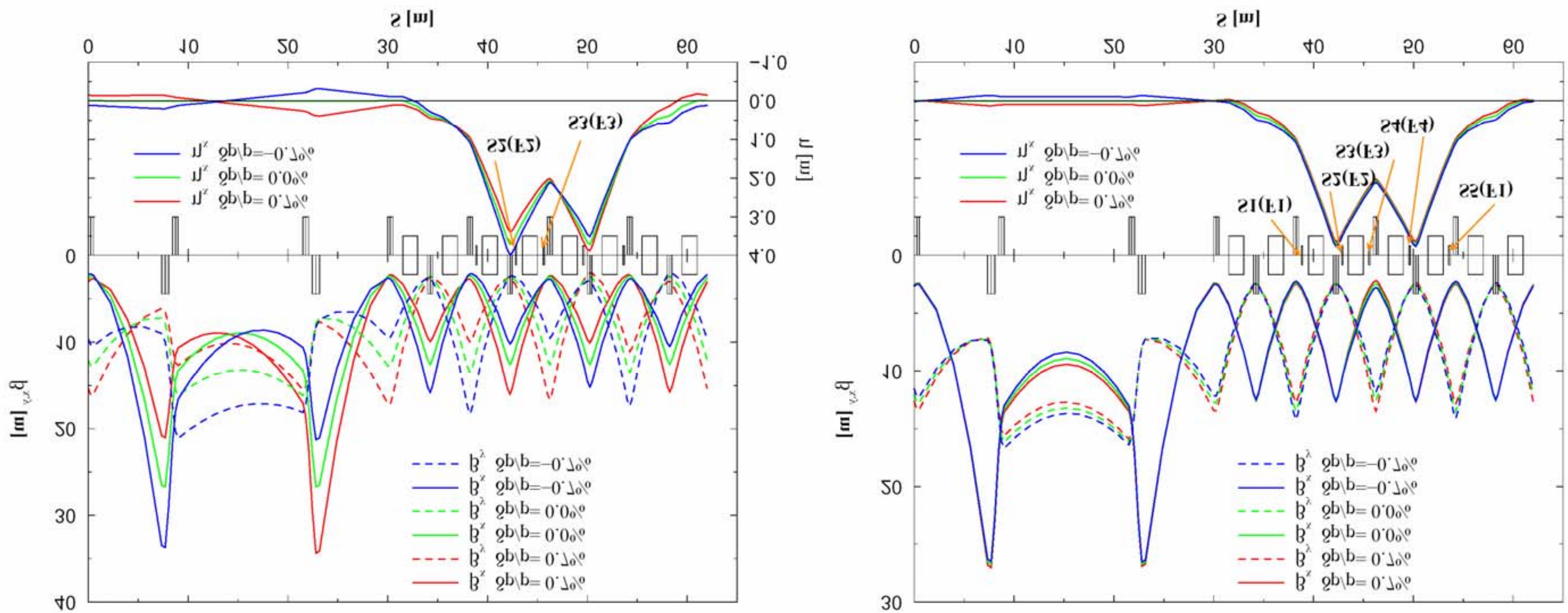
$$\xi_{x,y}^{tot} = -\frac{1}{4\pi}\oint \beta_{x,y}(s)(S(s)D_x(s) + k(s))ds$$

Chromaticity correction

- Introduce sextupoles in high-dispersion areas
- Tune them to achieve desired chromaticity
- Two families are able to control horizontal and vertical chromaticity
- Sextupoles introduce non-linear fields (chaotic motion)
- Sextupoles introduce tune-shift with amplitude
- Example:
  - The SNS ring has natural chromaticity of **–7**
  - Placing two sextupoles of length **0.3m** in locations where **β=12m**, and the dispersion **D=4m**
  - For getting **0** chromaticity, their strength should be

$$S = \frac{7 \cdot 4\pi}{12 \cdot 4 \cdot 2 \cdot 0.3} \approx 3\mathrm{m}^{-3}$$ or a gradient of **17.3 T/m²**

Two vs. four families for chromaticity correction



- Two families of sextupoles not enough for correcting off-momentum optics functions' distortion and second order chromaticity
- Solutions:
  - Place sextupoles accordingly to eliminate second order effects (difficult)
  - Use more families (4 in the case of of the SNS ring)
- Large optics function distortion for momentum spreads of ±**0.7%**,when using only two families of sextupoles
- Absolute correction of optics beating with four families

Non-linear imperfections and correction

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Non-linear effects and correction

- Kinematic effect
- Magnet fringe-fields
- Magnet imperfections
- Correction
  - Sextupole correction
  - Skew sextupole
  - Octupole correction
- Singe-particle diffusion
  - Dynamics aperture
  - Frequency maps

Kinematic effect

Kinematic non-linearity → high-order momentum terms in the expansion of the relativistic Hamiltonian

- Negligible in high energy colliders
- Noticeable in low-energy high-intensity rings

First-order tune-shift:

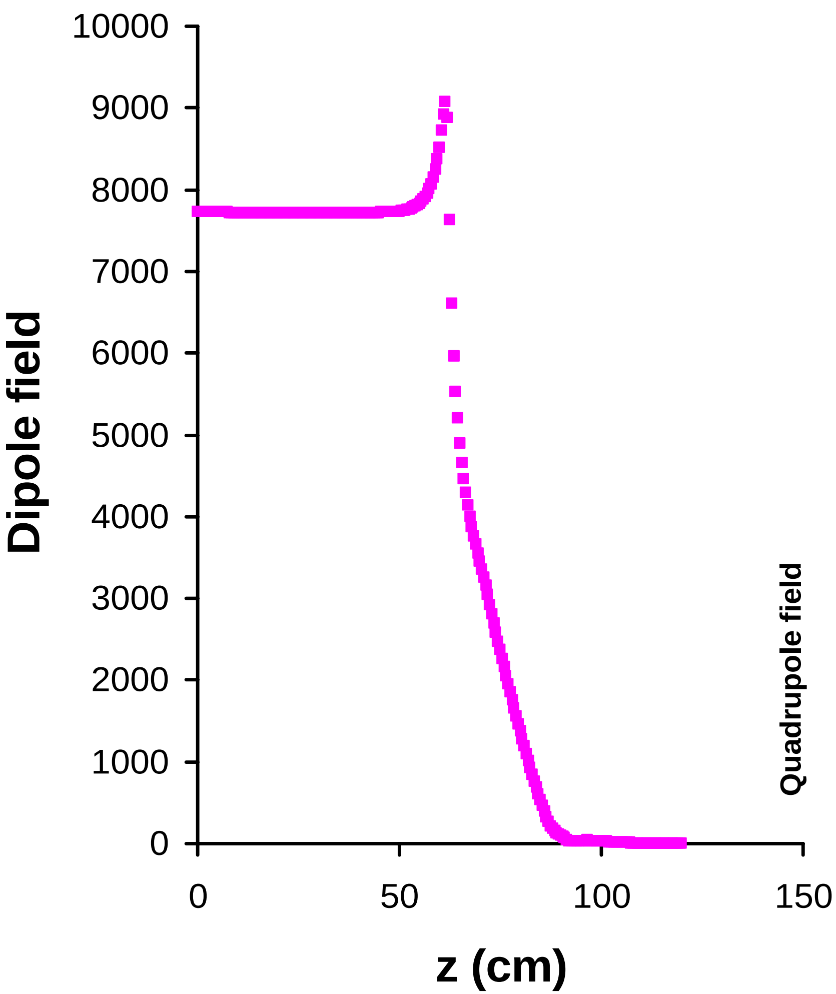
$$\delta Q_{x,y} = \frac{1}{2\pi}\sum_{k=2}^{\infty}\frac{(2k-3)!!}{2^k(2k)!!}\times\sum_{\lambda=0}^{k}\lambda\binom{2\lambda}{\lambda}\binom{k}{\lambda}\binom{2(k-\lambda)}{k-\lambda}J_{x,y}^{\lambda-1}J_{y,x}^{k-\lambda}G_{x,y}$$

where $G_{x,y} = \oint_{\text{ring}} \gamma_{x,y}^{\lambda}\gamma_{y,x}^{k-\lambda}ds$

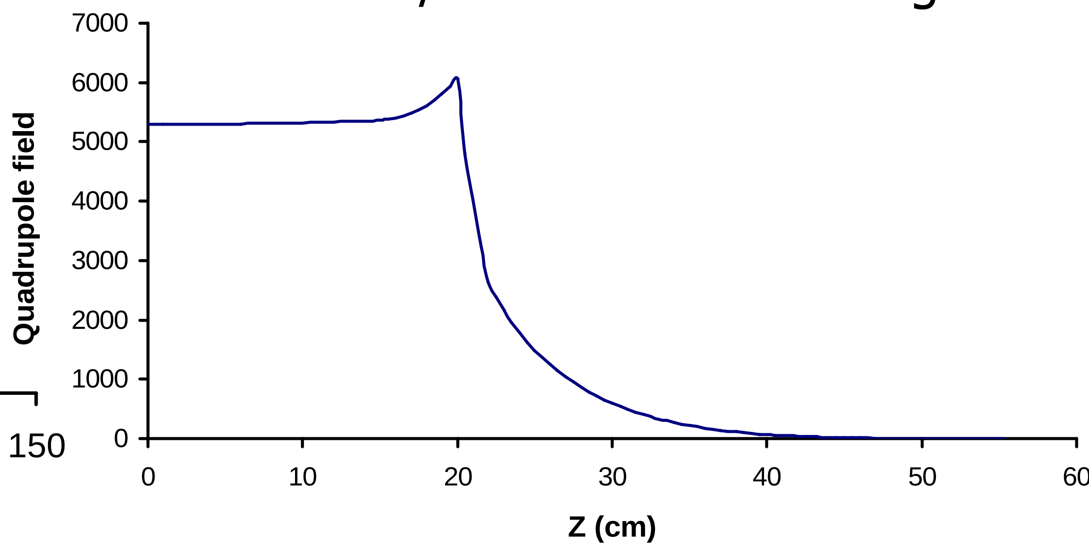
Leading order ⟶ octupole-type tune-shift

For the SNS ring, kinematic tune-shift is of the order of 0.001 @ 480 π.mm.mrad

Magnet fringe fields



- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big



Consider a 3D magnetic field

$$\mathbf{B}(x,y,z) = \nabla\Phi(x,y,z) = \frac{\partial\Phi}{\partial x}\mathbf{x} + \frac{\partial\Phi}{\partial y}\mathbf{y} + \frac{\partial\Phi}{\partial z}\mathbf{z} \ ,$$

where

$$\nabla^2\Phi(x,y,z) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \ .$$

Appropriate expansion:

$$\Phi(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{C}_{m,n}(z)\frac{x^n\, y^m}{n!\, m!} \ ,$$

By Laplace equation: $\mathcal{C}_{m+2,n} = -\mathcal{C}_{m,n+2} - \mathcal{C}_{m,n}^{[2]}$

3D multipole coefficients

The field components:

$$B_x(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{C}_{m,n+1}(z)\frac{x^n\, y^m}{n!\, m!}$$

$$B_y(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{C}_{m+1,n}(z)\frac{x^n\, y^m}{n!\, m!} ,$$

$$B_z(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{C}_{m,n}^{[1]}(z)\frac{x^n\, y^m}{n!\, m!}$$

The usual normal and skew multipole coefficients are:

$$b_n(z) = \mathcal{C}_{1,n}(z) = \left(\frac{\partial^n B_y}{\partial x^n}\right)(0,0,z)$$

$$a_n(z) = \mathcal{C}_{0,n+1}(z) = \left(\frac{\partial^n B_x}{\partial x^n}\right)(0,0,z) .$$

Note that $\mathcal{C}_{m,n} = \sum_{l=0}^{k}(-1)^k\binom{k}{l}\mathcal{C}_{m-2k,n+2k-2l}^{[2l]}$

3D field components

Consider two cases, for $m = 2k$ (even) or $m = 2k + 1$ (odd)

$$\mathcal{C}_{2k,n} = \sum_{l=0}^{k} (-1)^k \binom{k}{l} a^{[2l]}_{n+2k-2l-1} \text{ , for } n + 2k - 2l - 1 \geq 0$$

$$\mathcal{C}_{2k+1,n} = \sum_{l=0}^{k} (-1)^k \binom{k}{l} b^{[2l]}_{n+2k-2l} \text{ ,}$$

and finally the field components are

$$B_x(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{m} (-1)^m \binom{m}{l} \frac{x^n\, y^{2m}}{n!\,(2m)!} \left(b^{[2l]}_{n+2m+1-2l} \frac{y}{2m+1} + a^{[2l]}_{n+2m-2l} \right)$$

$$B_y(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{x^n\, y^{2m}}{n!\,(2m)!} \left[\sum_{l=0}^{m} \binom{m}{l} b^{[2l]}_{n+2m-2l} - \sum_{l=0}^{m+1} \binom{m+1}{l} a^{[2l]}_{n+2m+1-2l} \frac{y}{2m+1} \right] \text{ ,}$$

$$B_z(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{m} (-1)^m \binom{m}{l} \frac{x^n\, y^{2m}}{n!\,(2m)!} \left(b^{[2l+1]}_{n+2m-2l} \frac{y}{2m+1} + a^{[2l+1]}_{n+2m-1-2l} \right)$$

Dipole fringe field

Using the general z-dependent field expansion, for a straight dipole:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+2-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m}}{(2n)!(2m)!} \binom{m}{l} b_{2n+2m-2l}^{[2l]}$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m-2l}^{[2l+1]}$$

and to leading order:

$$\begin{aligned} B_x &= b_2 xy + O(4) \\ B_y &= b_0 - \frac{1}{2} b_0^{[2]} y^2 + \frac{1}{2} b_2 (x^2 - y^2) + O(4) \\ B_z &= y\, b_0^{[1]} + O(3) \end{aligned}$$

Dipole fringe to leading order gives a sextupole-like effect (vertical chromaticity)

Quadrupole fringe field

General field expansion for a quadrupole magnet:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} .$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l+1]}$$

and to leading order

$$\begin{aligned} B_x &= y\left[b_1 - \frac{1}{12}(3x^2+y^2)b_1^{[2]}\right] + O(5) \\ B_y &= x\left[b_1 - \frac{1}{12}(3y^2+x^2)b_1^{[2]}\right] + O(5) \\ B_z &= xyb_1^{[1]} + O(4) \end{aligned}$$

The quadrupole fringe to leading order has an octupole-like effect

Scaling law for magnet fringe fields

- Ratio between momentum components produced by fringe field over body contribution

If α small:

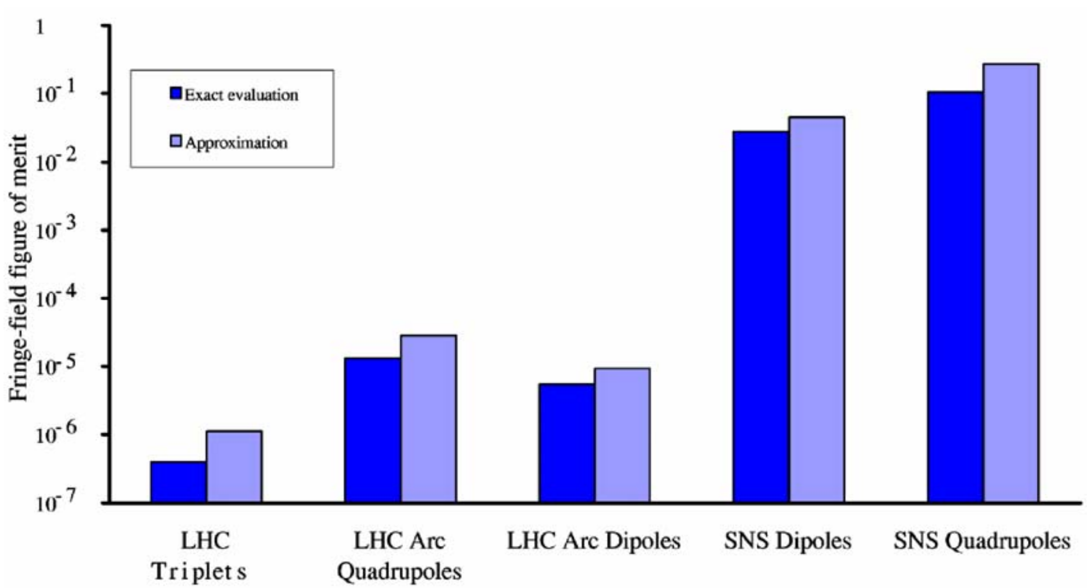
$$\frac{(\Delta p_\perp^f)_{rms}}{(\Delta p_\perp^b)_{rms}} \approx \frac{\epsilon_\perp}{L_{\rm eff}} \quad ,$$

where $\epsilon_\perp$ the rms beam transverse emittance.

When α or α_y large:

$$\frac{(\Delta p_\perp^f)_{rms}}{(\Delta p_\perp^b)_{rms}} \approx \alpha \frac{\epsilon_\perp}{L_{\rm eff}} \quad ,$$

where α the maximum of α_x or α_y.



Studying fringe fields effects

- Be sure that they are important for your machine (scaling law)
- Get an accurate magnet model or measurement
- Study dynamics
  - Integrating equations of motion
  - Build a non-linear map
    - Hard-edge approximation
    - Integrate magnetic field
    - Fit magnetic field with appropriate function (Enge function)
- Use your favorite non-linear dynamics tool to analyze the effect

Quadrupole fringe field in the SNS ring

The hard-edge Hamiltonian (Forest and Milutinovic 1988)

$$H_f = \frac{\pm Q}{12B\rho(1+\frac{\delta p}{p})}(y^3 p_y - x^3 p_x + 3x^2 y p_y - 3y^2 x p_x),$$

First order tune spread for an octupole:

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} a_{hh} & a_{hv} \\ a_{hv} & a_{vv} \end{pmatrix} \begin{pmatrix} 2J_x \\ 2J_y \end{pmatrix},$$

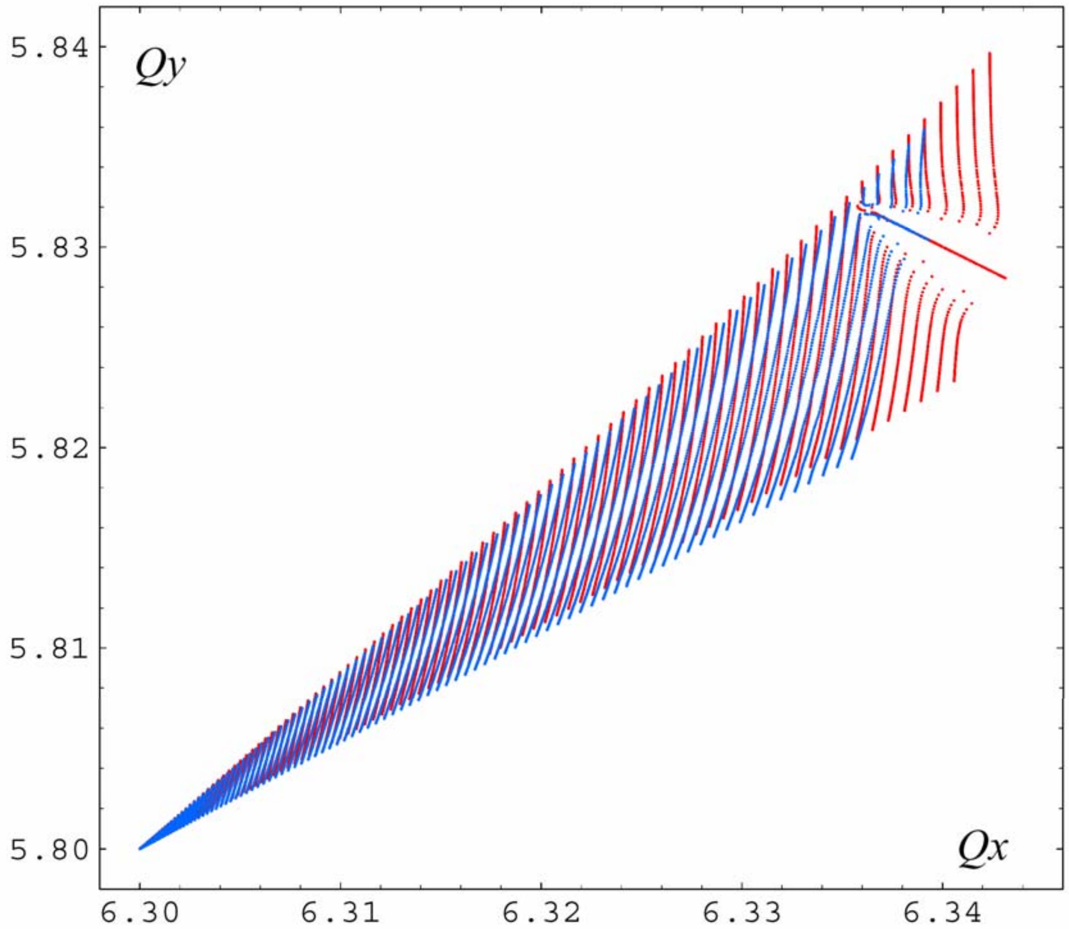
where the normalized anharmonicities are

$$a_{hh} = \frac{-1}{16\pi B\rho} \sum_i \pm Q_i \beta_{xi} \alpha_{xi},$$

$$a_{hv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i (\beta_{xi} \alpha_{yi} - \beta_{yi} \alpha_{xi}),$$

$$a_{vv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i \beta_{yi} \alpha_{yi}.$$

Quadrupole fringe field tune spread



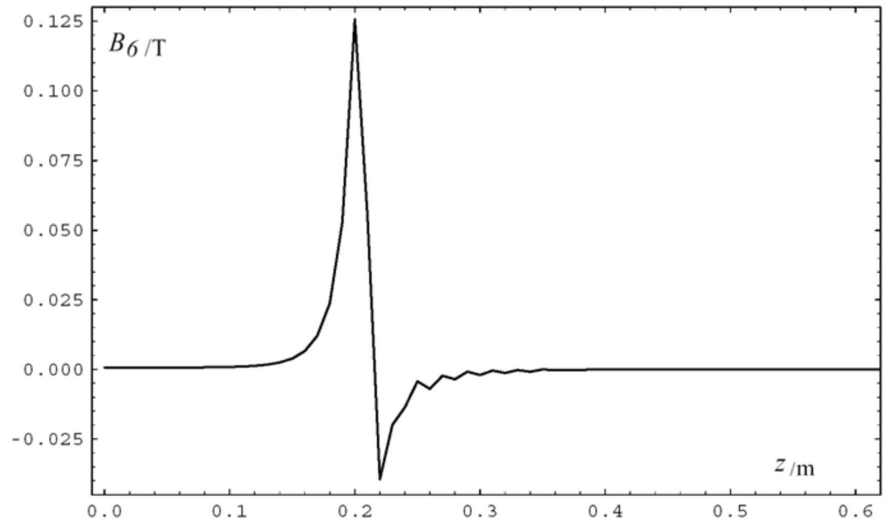
- Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field

Multipole errors

A perfect $2(n+1)$-pole magnet $\rightarrow \Phi(r, \theta, z) = \Phi(r, \frac{\pi}{n+1} - \theta, z)$ which gives

$n = (2j+1)(n+1) - 1$

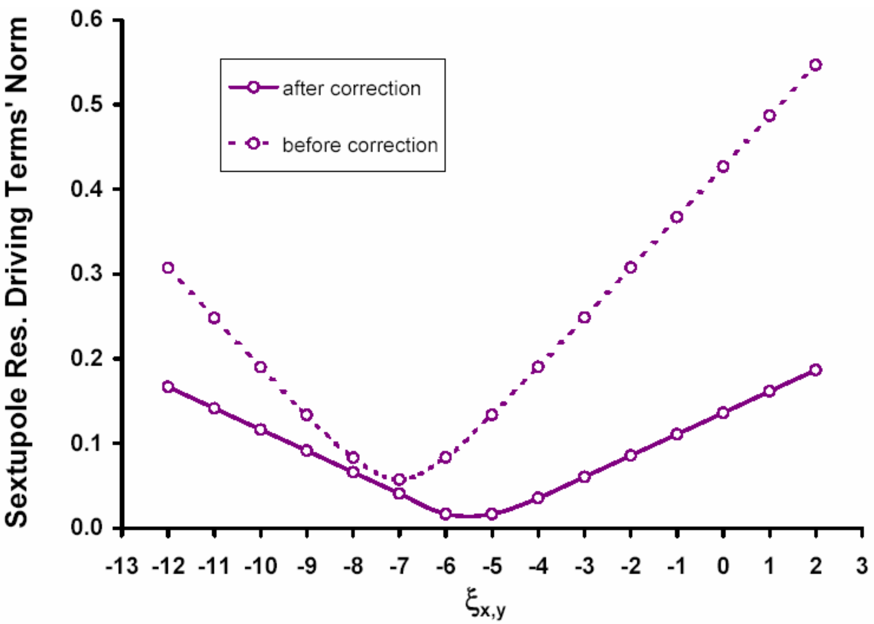
- Normal dipole $(n = 0) \longrightarrow b_{2j}$
- Normal quadrupole $(n = 1) \longrightarrow b_{4j+1}$
- Normal sextupole $(n = 2) \longrightarrow b_{6j+2}$



- All multi-pole components give suplementary non-linear effects that have to be quantified and corrected
- Most important the dodecapole component in a 21 cm quadrupole, with un-shaped ends. It is equal to 120.10^{-4} of the main quadrupole gradient.

Sextupole correction for the SNS ring

- Causes
  - Chromaticity sextupoles (small effect)
  - Sextupole errors in dipoles (10^{-4} level)
  - Dipole fringe-fields (small effect)

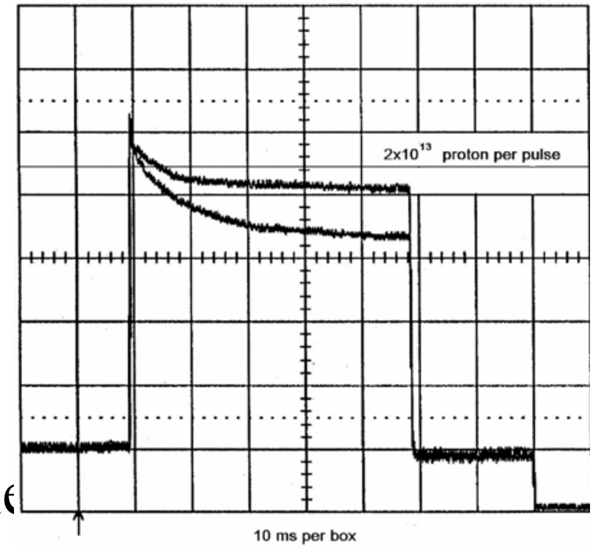


$$3Q_x = N \qquad Q_x \pm 2Q_y = N$$

- Effects
  - Zero first order tune-spread, octupole-like (linear in action) 2nd order
  - Excitation of normal sextupole resonances and
- Correction
  - Eight Sextupole correctors in symmetrical non-dispersive areas

Skew Sextupole correction for the SNS ring

- Causes
  - Chromaticity sextupoles roll
  - Dipole roll
  - Magnet multipoles
- Effects
  - Zero first order tune-spread, octupole 2nd order
  - Excitation of skew sextupole resonances $3Q_y = N$ and $2Q_x \pm Q_y = N$
- Correction
  - Skew sextupoles strings in the arc dipole correctors
  - Only connected 16 of them (at the beginning and end of the arc)
  - 8 families formed
  - Ability to correct resonant lines for all possible working points



Octupole correction for the SNS ring

- Causes
  - Quadrupole fringe-fields
  - Kinematic effect (small)
  - Octupole errors in magnets (10^{-4} level)
  - Sextupole, skew sextupole error give octupole-like tune-spread
- Effects
  - Tune-spread linear in action
  - Excitation of normal octupole resonances $4Q_{x,y} = N$ and $2Q_x \pm 2Q_y =$
- Correction
  - 8 octupole correctors at the end of the arcs, independently powered
  - Tune their strength to minimize resonance driving terms or tune-spread

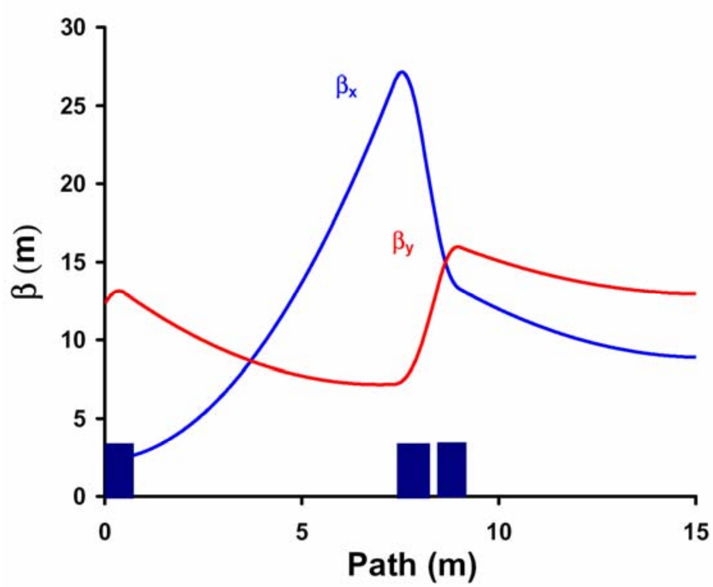
Octupole tune-spread correction

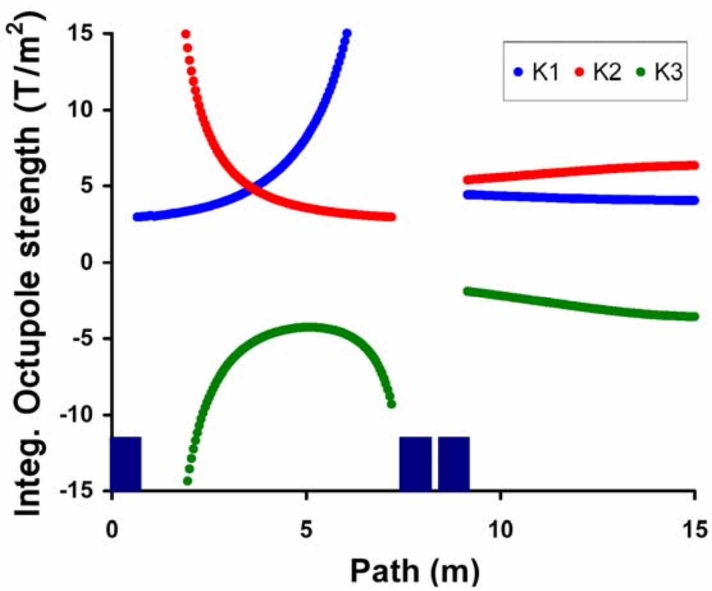
- The corrected anharmonisities become
- The area for a third octupole family is in the middle of the long straight section

$$A_{hh} = a_{hh} + \frac{3}{16\pi B\rho} \sum_j O_j \beta_{xj}^2,$$

$$A_{hv} = a_{hv} - \frac{6}{16\pi B\rho} \sum_j O_j \beta_{xj} \beta_{yj},$$

$$A_{vv} = a_{vv} + \frac{3}{16\pi B\rho} \sum_j O_j \beta_{yj}^2.$$





Error compensation in magnet design

Example: dodecapole in quadrupoles

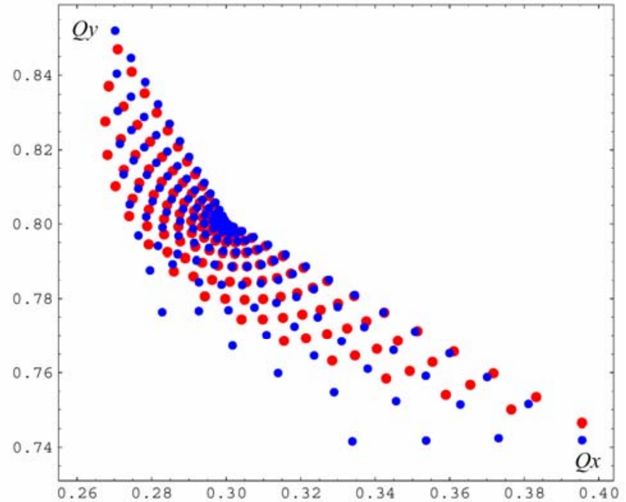
Tune-spread:

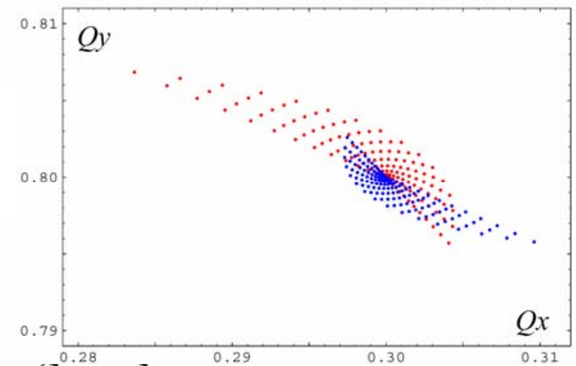
$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \sum_i \frac{b_{5i} Q_i}{8\pi B\rho} \mathcal{D}_i \begin{pmatrix} J_x^2 \\ J_x J_y \\ J_y^2 \end{pmatrix},$$

where $\mathcal{D}_i$ denotes the 3×2 matrix

$$\begin{pmatrix} \beta_{xi}^3 & -6\beta_{xi}^2\beta_{yi} & 3\beta_{xi}\beta_{yi}^2 \\ -3\beta_{xi}^2\beta_{yi} & 6\beta_{xi}\beta_{yi}^2 & -\beta_{yi}^3 \end{pmatrix}.$$

i.e. quadratic in the actions.

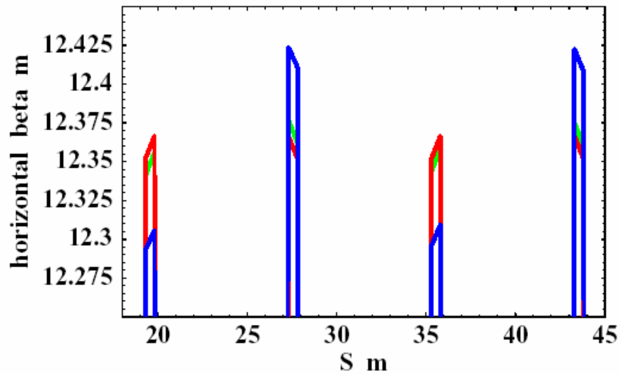




Method of correction $\longrightarrow$ Shape ends of the quadrupoles (local correction)

Magnet sorting

X beta-wave (21Q40 ITF sorting; 26Q40 & 30Q44, 30Q58 sorting of multipoles without ITF

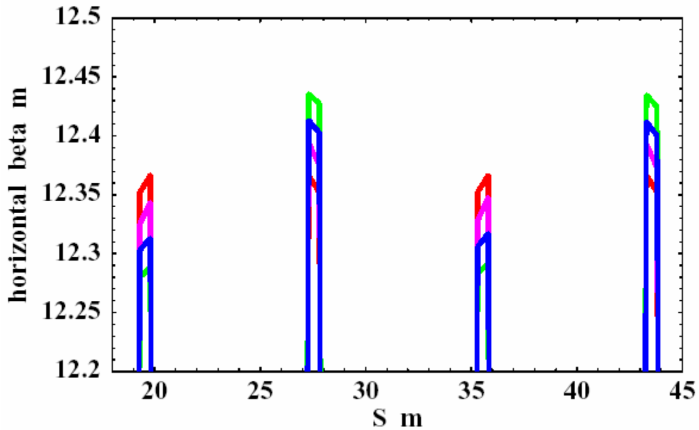


red- ideal
green – sorted 21Q40
blue – beta-wave
(mainly due to 30Q58)

- All measured quads=> 0.5% beta wave for and 10^{-3} tune shift.

- Two string of 8 ,21Q40
- One string of 12, 21Q40
- One string of 8, 26Q40
- One string of 8, 30Q58
- One sting of 8, 30Q44

X-beta wave at 1.3GeV with cut 30Q44

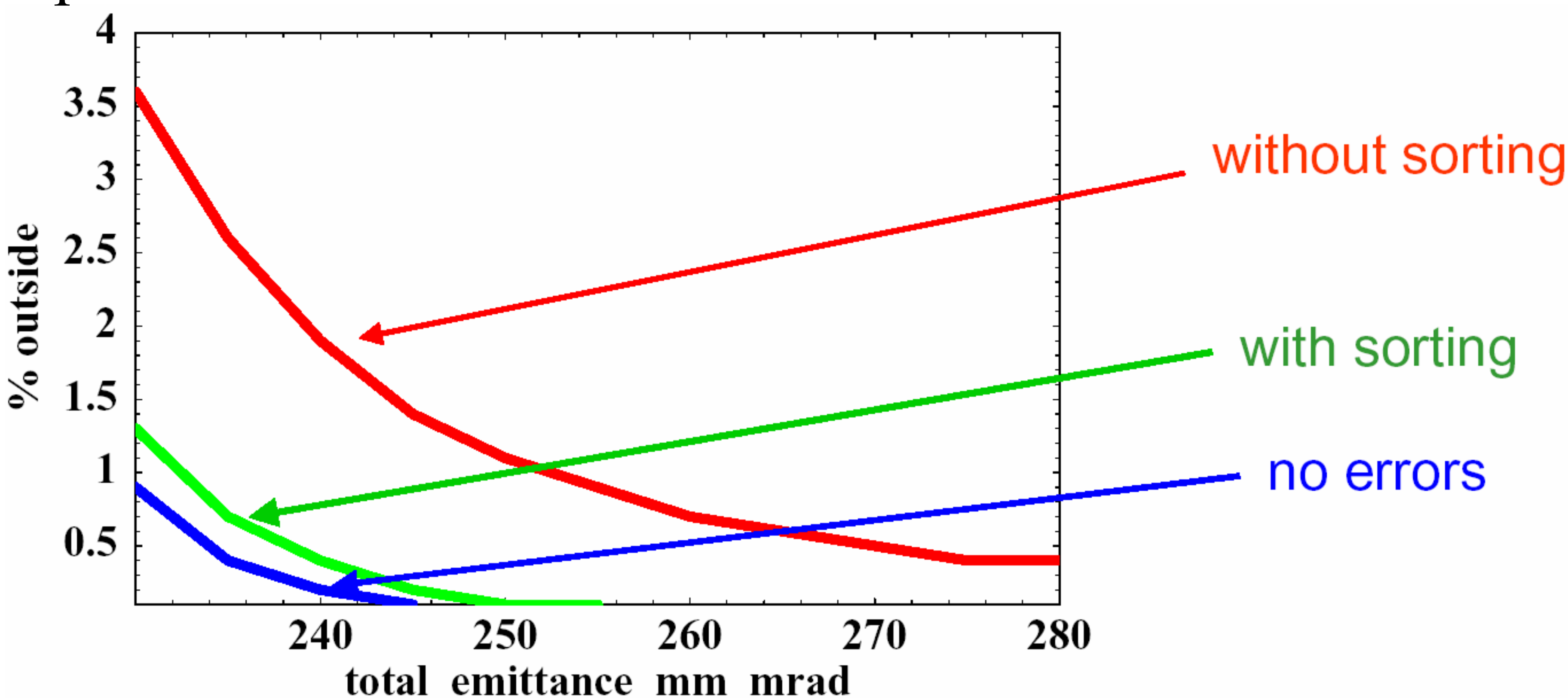


red – ideal
green – 2 magnets cut
blue – 1 magnet compensated
pink – both magnets compensated

All 21Q40 were sorted and 7 were shimmed. Three 26Q40 were shimmed and one re-aligned. All 30Q58 coils were shimmed, three 30Q58 iron was rotated

Sorting quadrupoles to minimize beam loss

- Sort magnets to minimize effects of dangerous resonances for working point (6.4,6.3)
- Balance out multi-pole errors based on a) total field b) phase advance



Single particle diffusion process

Three major types of diffusion :

a) Resonance overlapping: particles diffuse across resonance lines.

➔ FAST ~ 10^2 turns

b) Resonance streaming: particles diffuse along resonance lines.

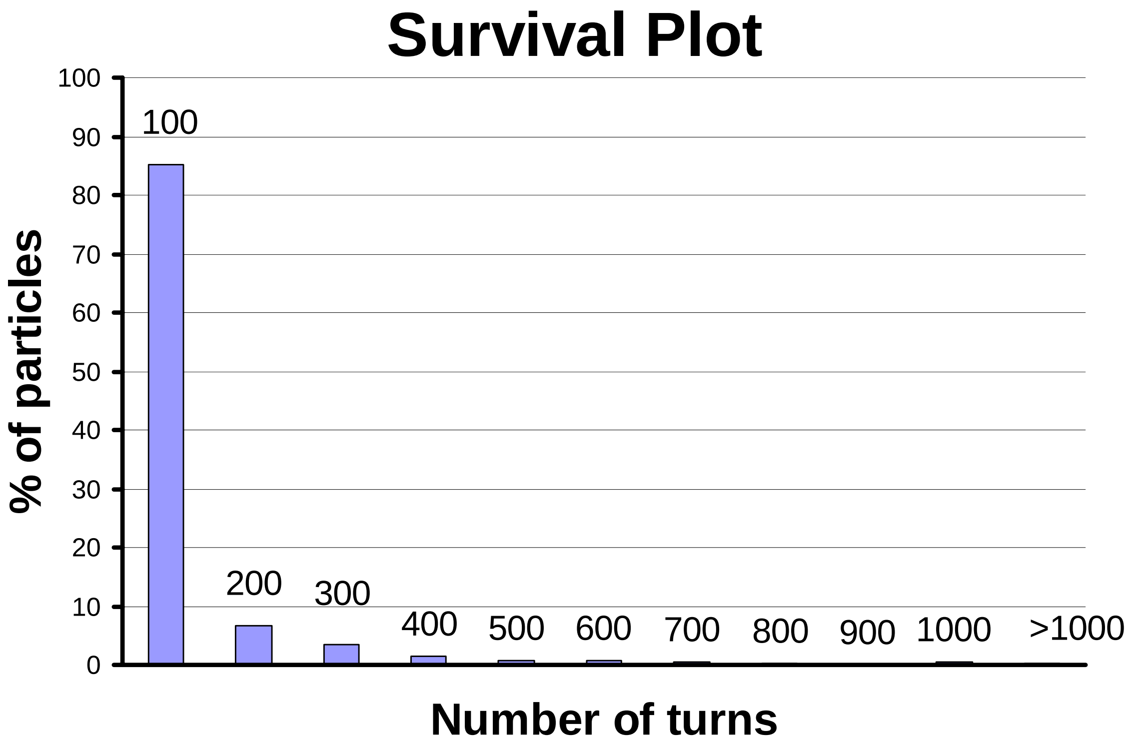
➔ SLOW ~ $\geq 10^4$ turns

c) Arnold diffusion: possibility of diffusion of particles in between the invariant tori of any slightly perturbed dynamical system (n>2).

➔ EXTREMELY SLOW ~ $\geq 10^7$ turns

- With the presence of magnetic errors **only** the machine performance cannot be compromised. BUT: Space-charge + chromaticity + errors + broken super-periodicity enhance particle diffusion
- Important complication:

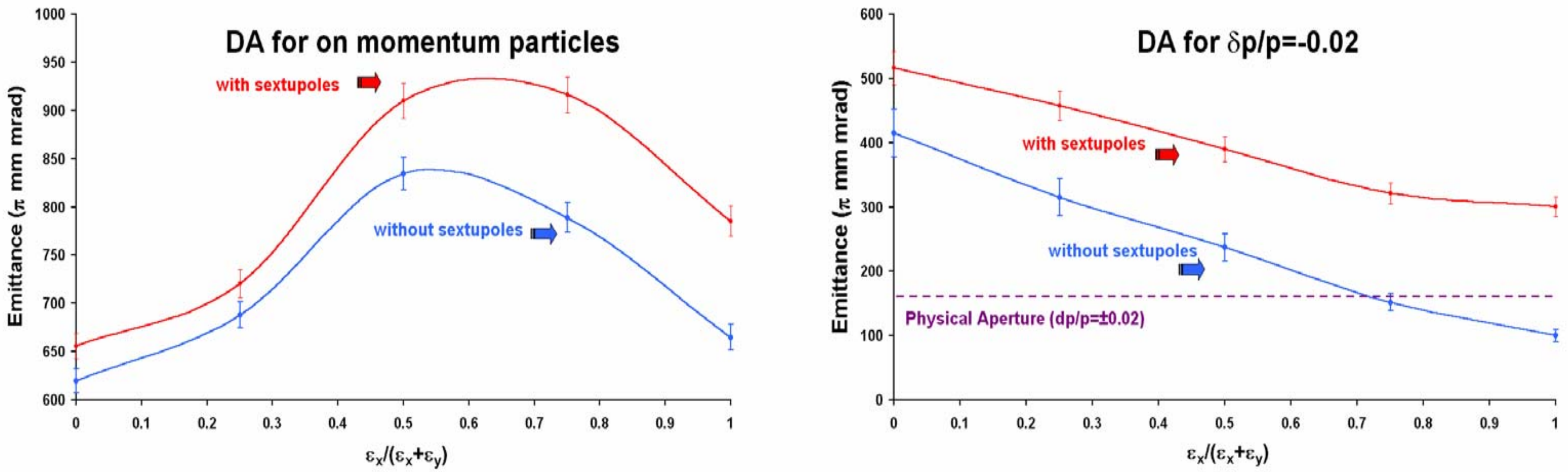
! The increase of the space-charge force due to beam accumulation shifts the particles in the frequency diagram



- Tracking ~ 1500 particles with amplitudes near the loss boundary
- 85% of particles are lost within the first 100 turns
- Less than 1% of lost particles survive for more than 1000 turns
- Fast diffusion due to resonance overlapping

Dynamic Aperture

Dynamic aperture tracking for on momentum particles (left) and for $\delta p/p$ = -0.02 (right), without (blue) and with (red) chromatic sextupoles

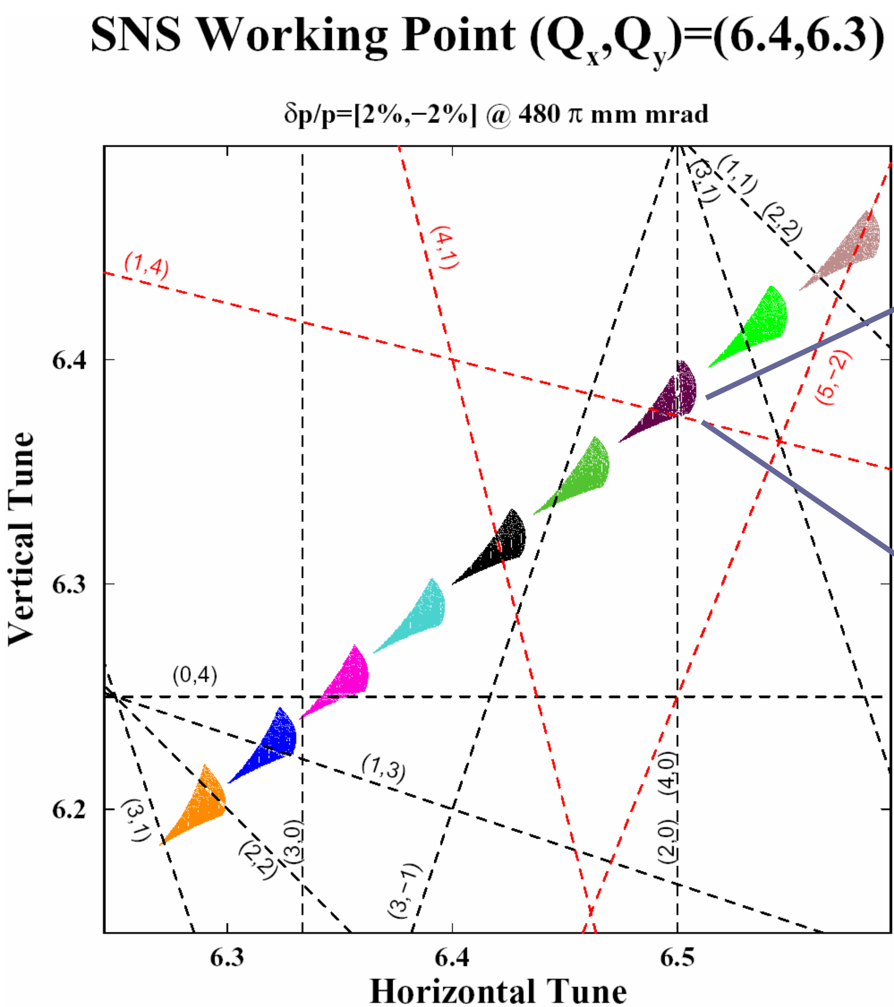


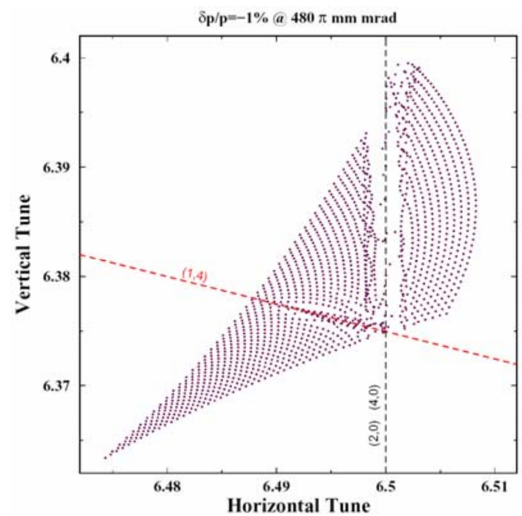
- **Drop of the DA without chromatic sextupoles in both cases**
- **Unacceptable drop below physical aperture for $\delta p/p$ = -0.02 (right)**

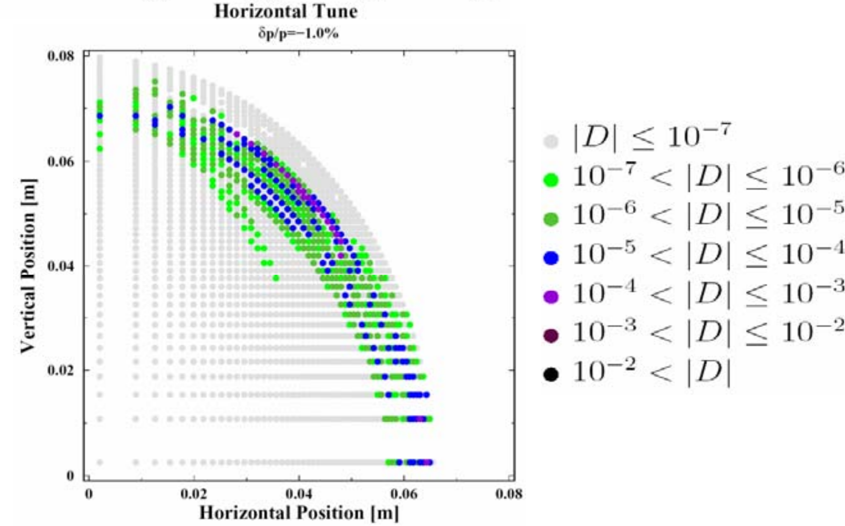
- Model includes
  - Magnet fringe-fields (5^{th} order maps)
  - Magnet systematic and random errors (10^{-4} level)
  - 4 working points, with and without chromaticity correction
  - No RF, no space-charge
- Single particle tracking using FTPOT module of UAL
  - 1500 particles uniformly distributed on the phase space up to 480 π mm mrad, with zero initial momentum, and 9 different momentum spreads (-2% to 2%)
  - 500 turns

$$\mathcal{F}_\tau : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (I_x, I_y)|_{p_x,p_y=0}, & \longrightarrow & (\nu_x, \nu_y) \end{array}$$

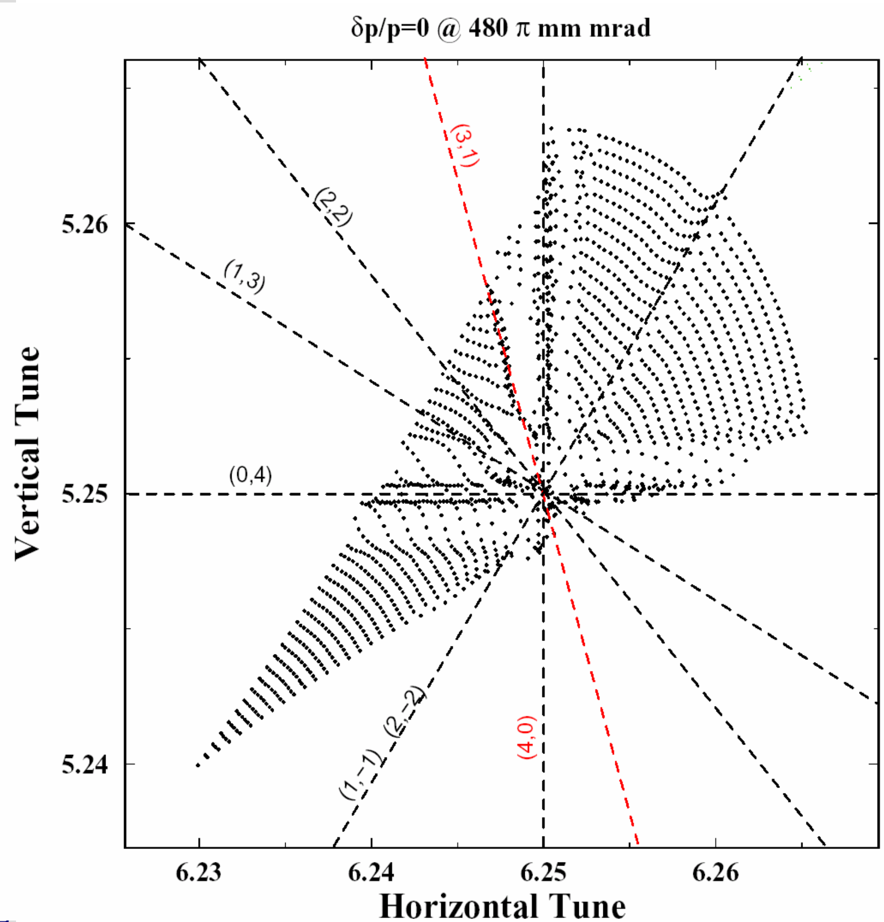
Working point (6.4,6.3)

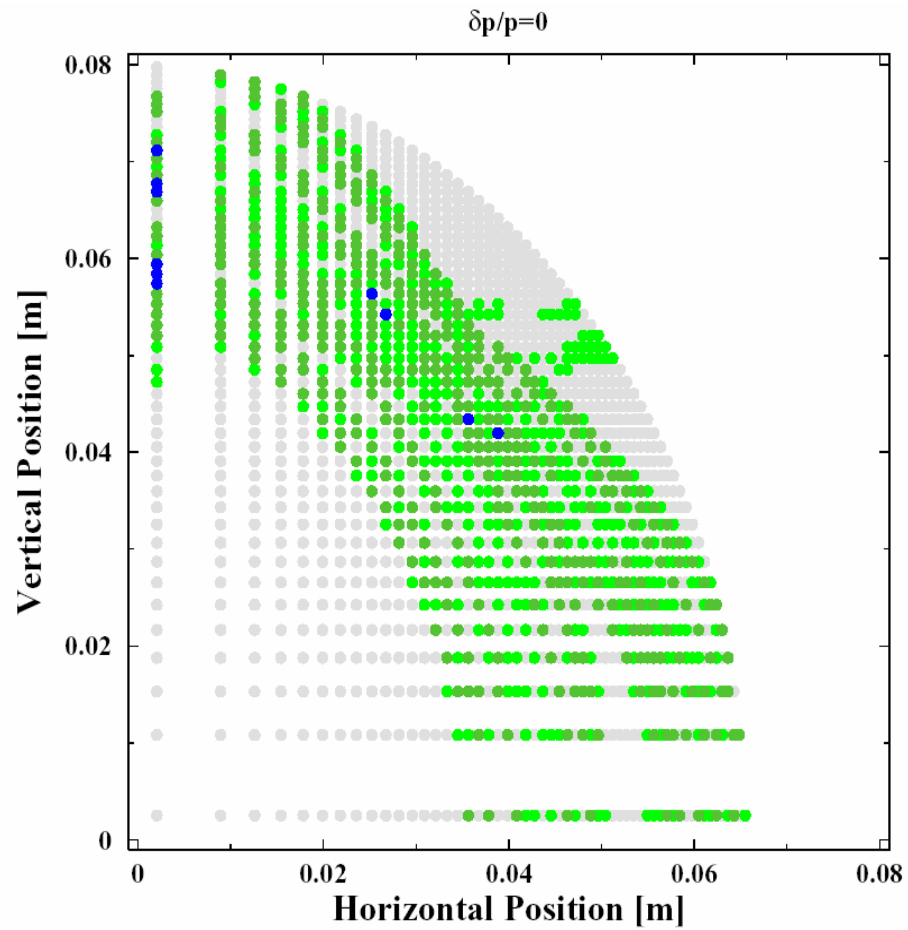






Working point (6.23,5.24)





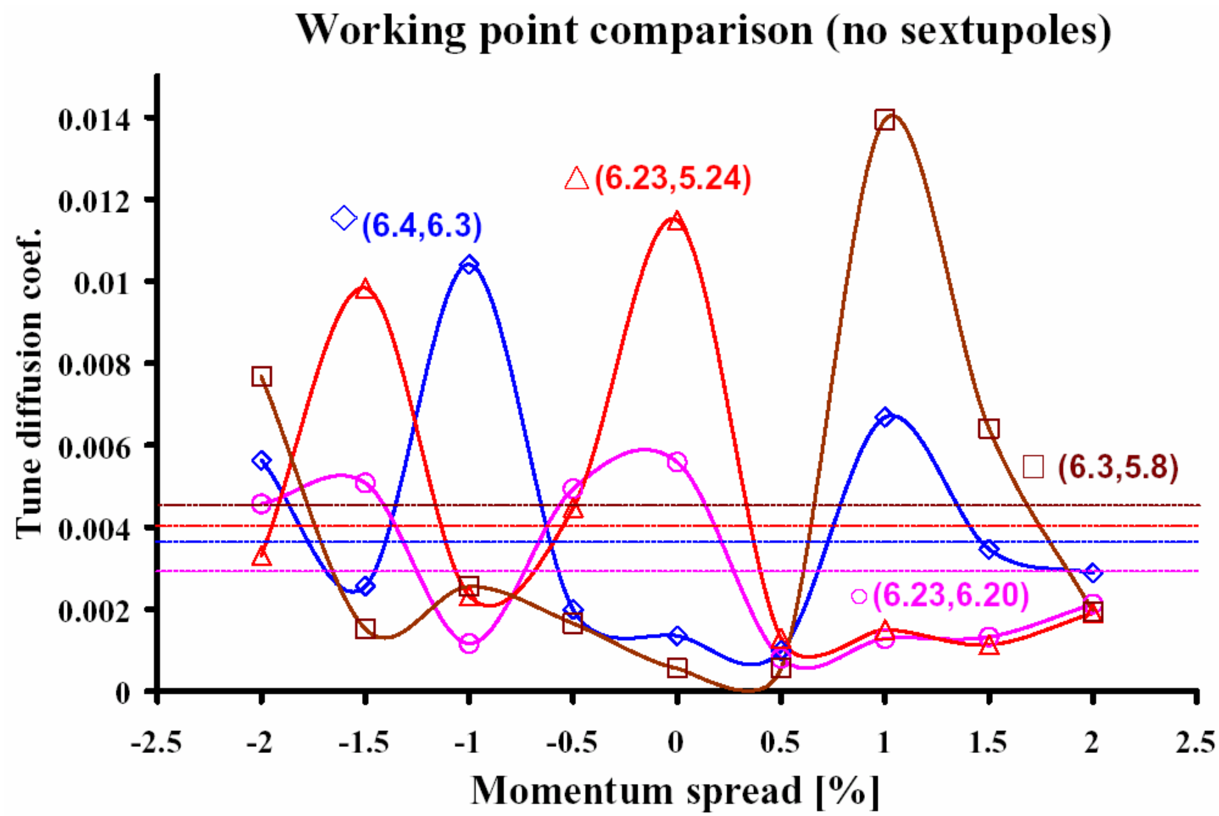
Resonance identification for (6.3,5.8)

Work. Point	δp/p (%)	Resonances	Possible Cause	Correction
(6.3,5.8)	-2.0	(2,-1)	a3 random error	Mag. Qual. + Skew Sext.
	-1.5	(3,3)	b6 error on quads	Mag. Qual.
	-1.0	(3,1) (1,3)	a4 random error	Mag. Qual.
	-0.5	(3,0) (1,2)	b3 error + dipole fringe fields	Mag.Qual. + Sextupole
	0.0			
	0.5			
	1.0	(1,1) (2,2)	Quad. fringe fields	Skew Quad. - Octupole
		(4,0) (2,-2) (0,4)	Quad. fringe fields	Octupole
		(3,-1) (1,-3)	a4 random error	Mag. Qual.
	1.5	(1,1) (2,2)	Quad. fringe fields	Skew Quad. - Octupole
		(4,0) (2,-2) (0,4)	Quad. fringe fields	Octupole
		(1,-3)	a4 random error	Mag. Qual.
	2.0			

Working Point Comparison

Tune Diffusion quality factor

$$D_{QF} = \left\langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$



Correction packages

Baseline	Quantity	Powering	Justification
Dipole	52 (+2)	Individual	Injection dump dipoles
TRIM Quadrupoles	52	28 families	Beta beating correction due to lattice symmetry breaking
Skew Quadrupoles	16	Individual	Coupling correction
High-Field Sextupoles	20	4 families	Correction of large chromatic effect
Normal Sextupoles	8	Individual	Sextupole resonance correction due to sextupole errors and octupole feed-down
Skew Sextupoles	16	8 families	Skew sextupole resonance correction (AGS booster)
Octupoles	8	Individual	Octupole resonance correction due to quadrupole fringe-fields