Linear imperfections and correction

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Outline

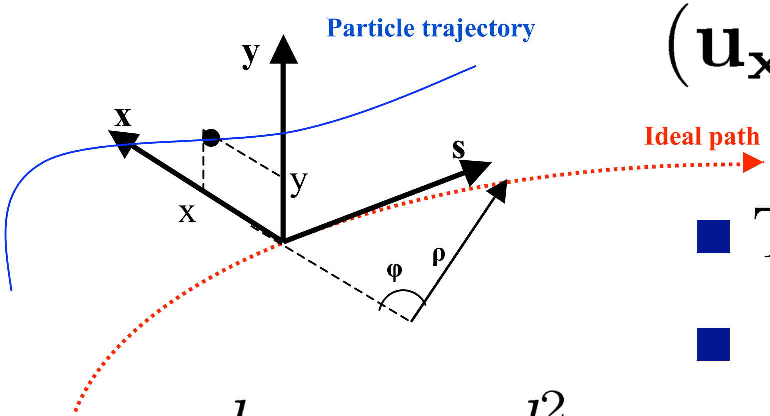
- Steering error and closed orbit distortion
- Gradient error and beta beating correction
- Linear coupling and correction
- Chromaticity

References

- O. Bruning, Linear imperfections, CERN Accelerator, 2006.
- J. Rossbach and P. Schmuser, Basic course on accelerator optics, CERN Accelerator School, 1992.
- H. Wiedemann, Particle Accelerator Physics I, Springer, 1999.
- K.Wille, The physics of Particle Accelerators, Oxford University Press, 2000.

Coordinate system

- Cartesian coordinates not useful to describe motion in an accelerator
- Instead a system following an ideal path along the accelerator is used (**Frenet** reference system)



$$(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z}) \rightarrow (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_s})$$

- The curvature vector is $\mathbf{k} = -\frac{d^2\mathbf{s}}{ds^2}$
- From Lorentz equation

$$\frac{d\mathbf{p}}{dt} = m\gamma\frac{d^2\mathbf{s}}{dt^2} = m\gamma v_s^2\frac{d^2\mathbf{s}}{ds^2} = -m\gamma v_s^2\mathbf{k} = q|\mathbf{v} \times \mathbf{B}|$$

- The ideal path is defined by $\mathbf{k} = -\frac{q}{p}|\frac{\mathbf{v}}{v_s} \times \mathbf{B}|$

Beam guidance

- Consider uniform magnetic field ***B*** in the direction perpendicular to particle motion. From the ideal trajectory and after considering that the transverse velocities $v_x \ll v_s, v_y \ll v_s$, the radius of curvature is

$$\frac{1}{\rho} = |k| = |\frac{q}{p}B| = |\frac{q}{\beta E_{tot}}B|$$

- The **cyclotron** or **Larmor frequency** $\omega_L = |\frac{qc^2}{E_{tot}}B|$

- We define the **magnetic rigidity** $|B\rho| = \frac{p}{q}$

- In more practical units $\beta E_{tot}[GeV] = 0.2998|B\rho|[Tm]$

- For ions with charge multiplicity ***Z*** and atomic number ***A***, the energy per nucleon is

$$\beta \bar{E}_{tot}[GeV/u] = 0.2998\frac{Z}{A}|B\rho|[Tm]$$

Dipoles

- Consider an accelerator ring for particles with energy ***E*** with ***N*** dipoles of length ***L***

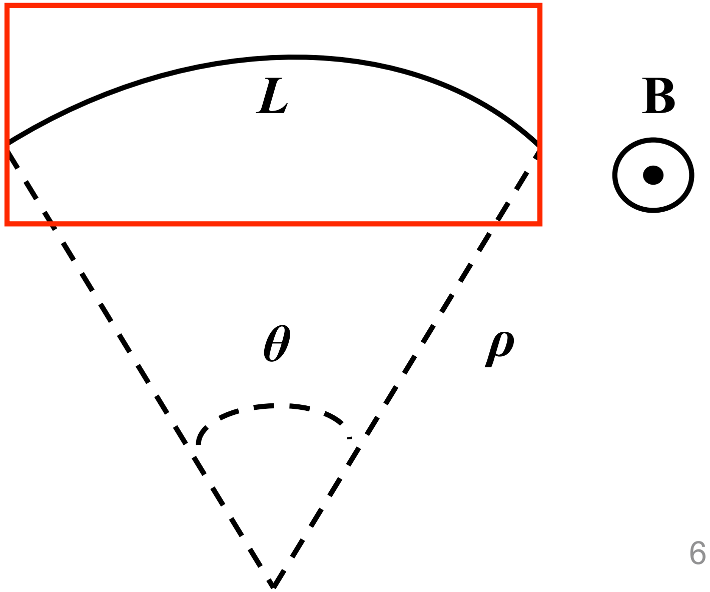
- **Bending angle** $$\theta = \frac{2\pi}{N}$$

- **Bending radius** $$\rho = \frac{L}{\theta}$$

- **Integrated dipole strength**

$$BL = \frac{2\pi}{N}\frac{\beta E}{q}$$

- Comments:
  - By choosing a dipole field, the dipole length is imposed and vice versa
  - The higher the field, shorter or smaller number of dipoles can be used
  - Ring circumference (cost) is influenced by the field choice



Equations of motion – Linear fields

- Consider *s*-dependent fields from dipoles and normal quadrupoles

$$B_y = B_0(s) - g(s)x \ , \quad B_x = -g(s)y$$

- The total momentum can be written $P = P_0(1 + \frac{\Delta P}{P})$
- With magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and normalized gradient

$k(s) = \frac{g(s)}{B_0\rho}$ the equations of motion are

$$x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right) x = \frac{1}{\rho(s)} \frac{\Delta P}{P}$$
$$y'' + k(s)\ y = 0$$

- Inhomogeneous equations with *s*-dependent coefficients
- The term $1/\rho^2$ corresponds to the dipole **week focusing**
- The term $\Delta P/(P\rho)$ represents **off-momentum** particles

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Hill's equations

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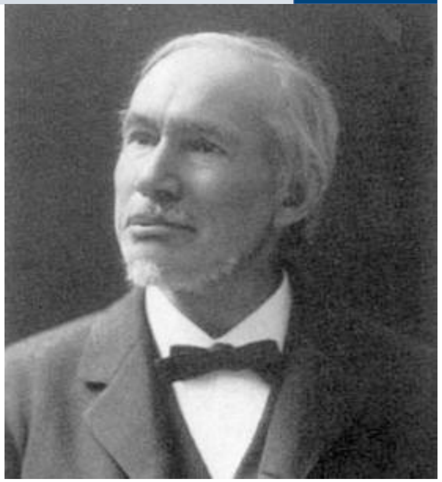


George Hill

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}$$

with $K_x(s) = -\left(k(s) - \dfrac{1}{\rho(s)^2}\right)$, $K_y(s) = k(s)$

- **Hill's equations of linear transverse particle motion**
- Linear equations with *s*-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring (or in transport line with symmetries), coefficients are periodic $K_x(s) = K_x(s+C)$, $K_y(s) = K_y(s+C)$
- Not straightforward to derive analytical solutions for whole accelerator

Betatron motion

- The on-momentum linear betatron motion of a particle is described by

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \psi_0)$$

with $\alpha,\ \beta,\ \gamma$ the twiss functions $\alpha(s) = -\frac{\beta(s)'}{2},\quad \gamma = \frac{1+\alpha(s)^2}{\beta(s)}$

ψ the **betatron phase** $\psi(s) = \int \frac{ds}{\beta(s)}$

and the **beta function** β is defined by the **envelope equation**

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$$

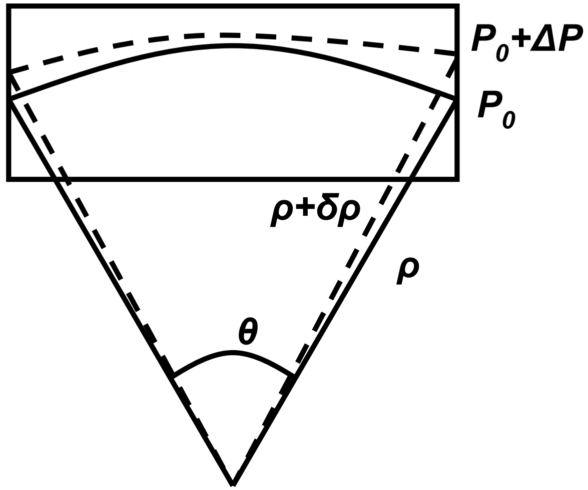
- By differentiation, we have that the **angle** is

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s) + \psi_0) + \alpha(s)\cos(\psi(s) + \psi_0)\right)$$

Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum P_0
- What happens for off-momentum particles, i.e. particles with momentum $P_0+\Delta P$?
- Consider a dipole with field B and bending radius ρ
- Recall that the magnetic rigidity is $B\rho = \frac{P_0}{q}$ and for off-momentum particles

$$B(\rho + \Delta\rho) = \frac{P_0 + \Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$



- Considering the effective length of the dipole unchanged

$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Off-momentum particles get different deflection (different orbit)

$$\boxed{\Delta\theta = -\theta\frac{\Delta P}{P_0}}$$

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way, the equations of motion are split in two parts

$$\begin{aligned} x_H'' + K_x(s)x_H &= 0 \\ x_I'' + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- The **dispersion function** can be defined as $D(s) = \dfrac{x_I(s)}{\Delta P/P}$
- The dispersion equation is

$$D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}$$

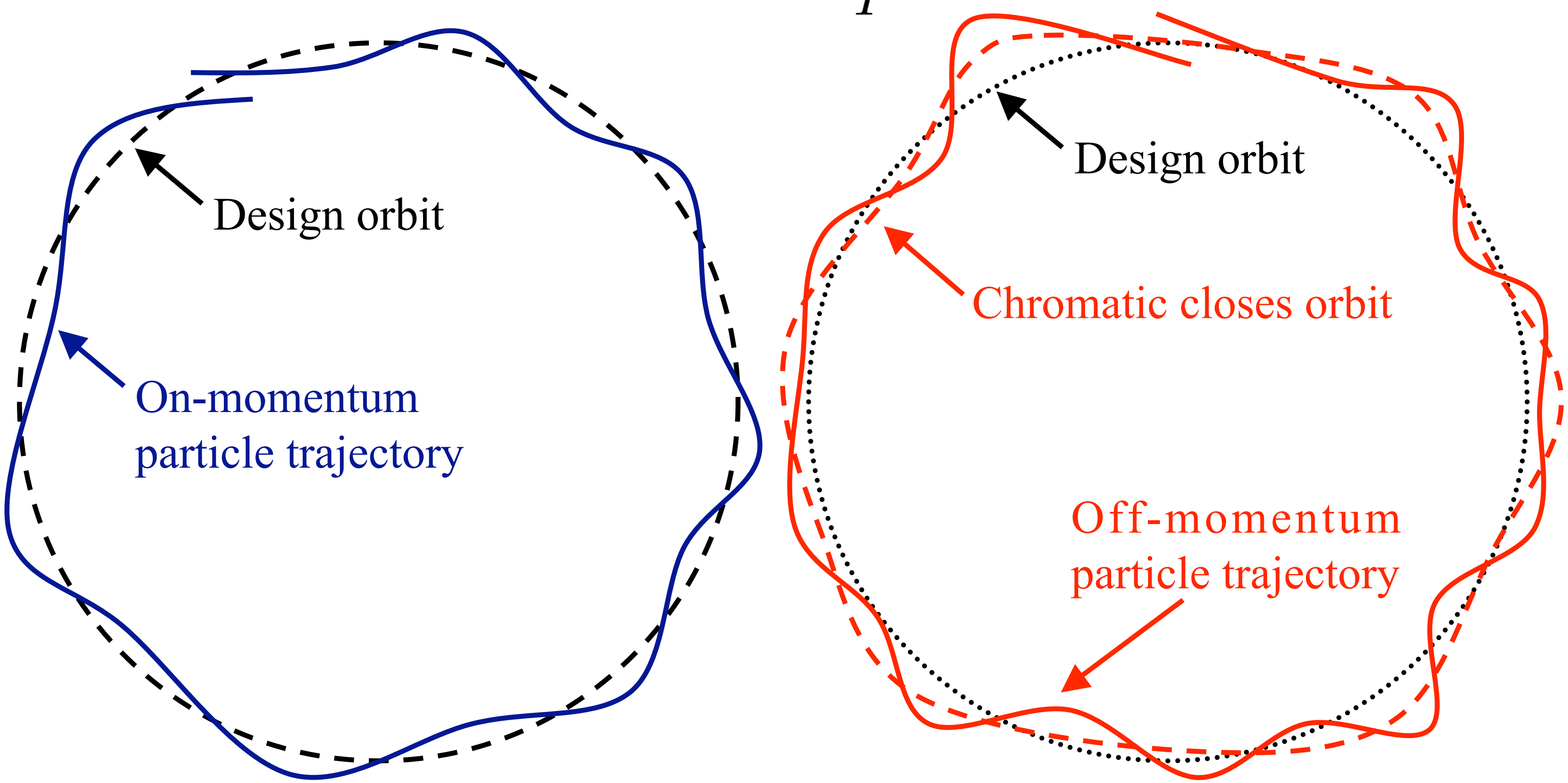
Beam stability in storage rings

- Beam orbit stability very critical
  - Injection and extraction efficiency of synchrotrons
  - Stability of collision point in colliders
  - Stability of the synchrotron light spot in the beam lines of light sources
- Consequences of orbit distortion
  - Miss-steering of beams, modification of the dispersion function, resonance excitation, aperture limitations, lifetime reduction, coupling of beam motions, modulation of lattice functions, poor injection efficiency
- Long term Causes (Years - months)
  - Ground settling, season changes, diffusion,
- Medium - Days/Hours,
  - Sun and moon, day-night variations (thermal), rivers, rain, wind, refills and start-up, sensor motion, drift of electronics, local machinery, filling patterns
- Short (Minutes/Seconds)
  - Ground vibrations, power supplies, injectors, experimental magnets, air conditioning, refrigerators/compressors, water cooling

Closed orbit

- Design orbit defined by main dipole field
- On-momentum particles oscillate around design orbit
- Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit
- Distance from the design orbit depends linearly with momentum spread and dispersion

$$x_D = D(s)\frac{\Delta P}{P}$$



Closed orbit distortion

- Causes
  - Dipole field errors
  - Dipole misalignments
  - Quadrupole misalignments
    - Consider the displacement of a particle **δx** from the ideal orbit . The vertical field is

$$B_y = G\bar{x} = G(x+\delta x) = \underbrace{Gx}_{\text{quadrupole}} + \underbrace{G\delta x}_{\text{dipole}}$$

- Remark: Dispersion creates a closed orbit distortion for off-momentum particles $\delta x = D(s)\frac{\delta p}{p}$
- Effect of orbit errors in any multi-pole magnet

$$B_y = b_n \bar{x}^n = b_n (x+\delta x)^n = b_n (\underbrace{x^n}_{\textbf{2(n+1)-pole}} + \underbrace{n\delta x x^{n-1}}_{\textbf{2n-pole}} + \underbrace{\frac{n(n-1)}{2}(\delta x)^2 x^{n-2}}_{\textbf{2(n-1)-pole}} + \cdots + \underbrace{(\delta x)^n}_{\textbf{dipole}})$$

- **Feed-down**

Effect of single dipole kick

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int \frac{ds}{\beta(s)}$$

- The Hill's equations are written $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator $\mathcal{U} = \mathcal{U}_0 \cos(\nu\phi)$
- Consider a single dipole kick $\delta u'(\pi) = \frac{\delta(Bl)}{B\rho}$ at **$\varphi=\pi$**

- Then $\mathcal{U}'(\pi) = -\mathcal{U}_0\nu\sin(\pi\nu) = \frac{d\mathcal{U}}{d\phi}\Big|_{\phi=\pi} = \frac{d\mathcal{U}}{ds}\frac{ds}{d\phi}\Big|_{s=k} = \frac{d\mathcal{U}}{ds}\Big|_{s=k}\beta(k)\nu = \sqrt{\beta(k)}\frac{du}{ds}\Big|_{s=k}$

and $\mathcal{U}_0 = \frac{\sqrt{\beta(k)}}{2|\sin(\nu\pi)|}\delta u'(\pi)$ with $\frac{\delta u'(\pi)}{2} = \frac{du}{ds}\Big|_{s=k} = \frac{\delta(Bl)}{2B\rho}$

and in the old coordinates

$$u(s) = \sqrt{\beta(s)}\mathcal{U}_0\cos(\nu\phi(s)) = \underbrace{\frac{\sqrt{\beta(s)\beta(k)}}{2\sin(\pi\nu)}}\frac{\delta(Bl)}{B\rho}\cos(\nu\phi(s))$$

Maximum distortion amplitude

Closed orbit distortion

Horizontal-vertical orbit distortion (Courant and Snyder 1957)

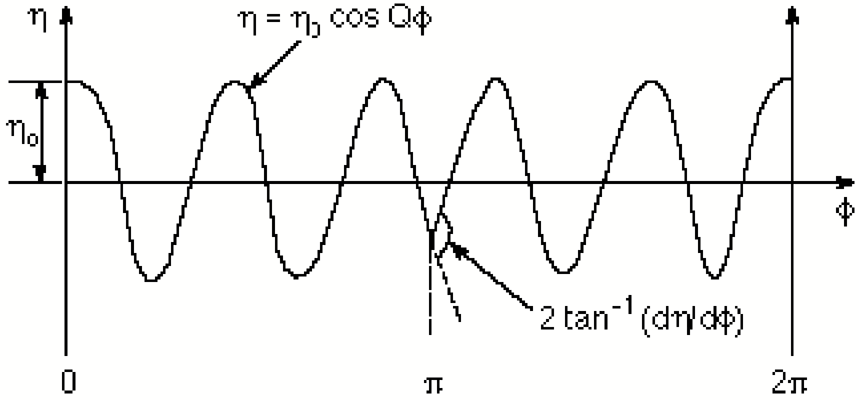
$$\delta_{x,y}(s) = -\frac{\sqrt{\beta_{x,y}}}{2\sin(\pi Q_{x,y})}\int_s^{s+C}\frac{\Delta B(\tau)}{B\rho}\sqrt{\beta_{x,y}}\cos(|\pi Q_{x,y}+\psi_{x,y}(s)-\psi_{x,y}(\tau)|)d\tau$$

with $\Delta B(\tau)$ the equivalent magnetic field error at $s=\tau$.

Approximate errors as delta functions in n locations:

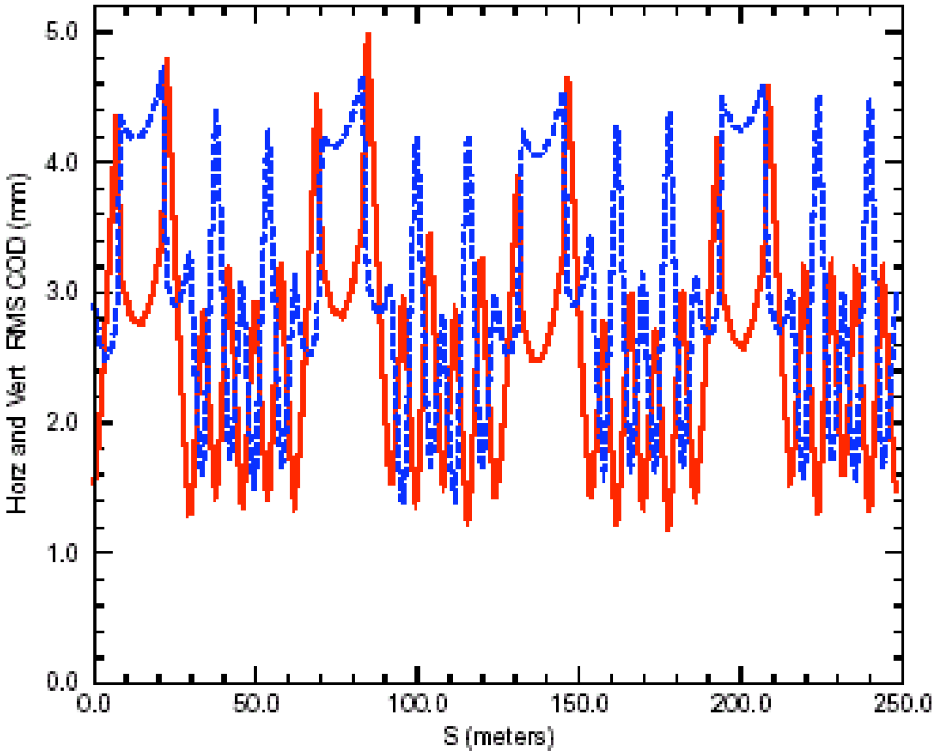
$$\delta_{x,y;i} = -\frac{\sqrt{\beta_{x,y;i}}}{2\sin(\pi Q_{x,y})}\sum_{j=i+1}^{i+n}\phi_{x,y;j}\sqrt{\beta_{x,y;j}}\cos(|\pi Q_{x,y}+\psi_{x,y;i}-\psi_{x,y;j}|)$$

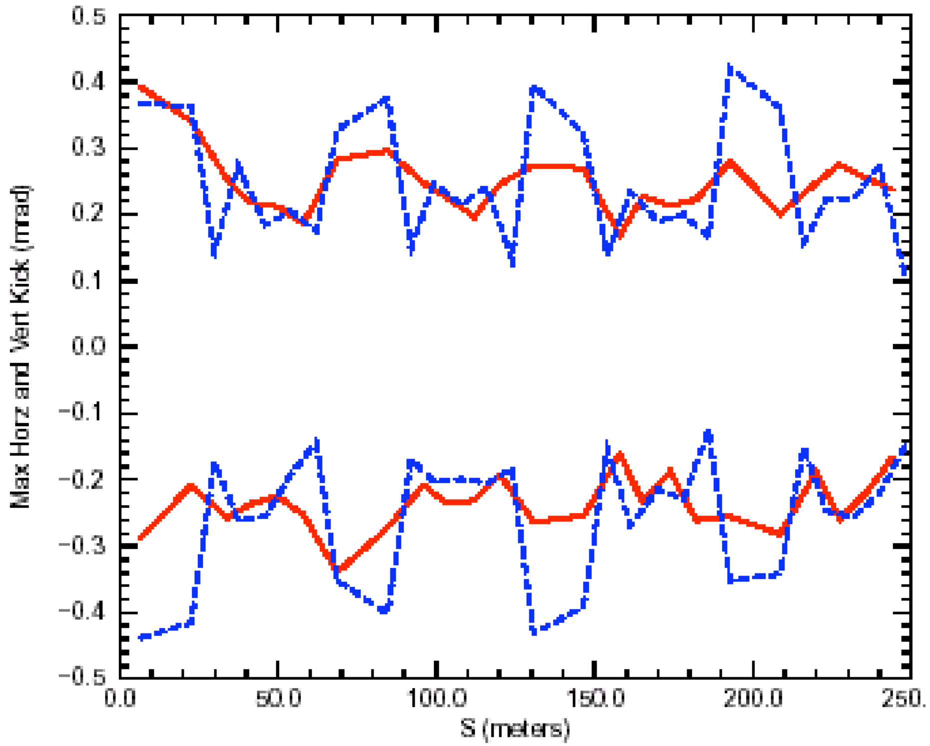
with $\phi_{x,y;j}$ kick produced by jth element



- $\phi_j = \frac{\Delta B_j L_j}{B\rho} \rightarrow$ dipole field error
- $\phi_j = \frac{B_j L_j \sin\theta_j}{B\rho} \rightarrow$ dipole roll
- $\phi_j = \frac{G_j L_j \Delta x, y_j}{B\rho} \rightarrow$ quadrupole displacement

Example: Orbit distortion for the SNS ring





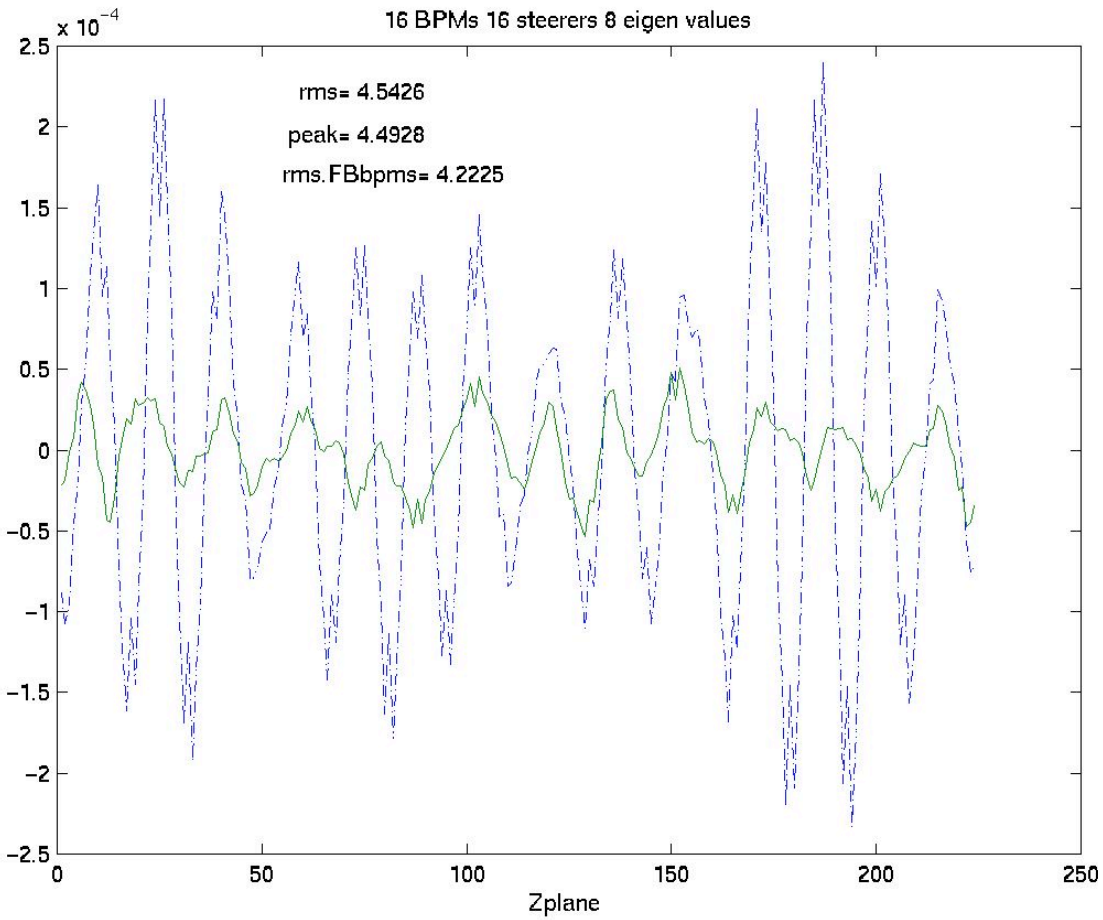
- In the SNS accumulator ring, the beta function is **6m** in the dipoles and **30m** in the quadrupoles.
- Consider dipole error of **δy'=1mrad**
- The tune is 6.2
- The maximum orbit distortion in the dipoles is $y_0 = \frac{\sqrt{6 \cdot 6}}{2\sin(6.2\pi)} \cdot 10^{-3} \approx 5\text{mm}$
- For quadrupole displacement with **0.5mrad** error the distortion is $y_0 \approx 2.5\text{cm}$!!!

Example: Orbit distortion in ESRF storage ring

- In the ESRF storage ring, the beta function is **1.5m** in the dipoles and **30m** in the quadrupoles.
- Consider dipole error of **δy'=1mrad**
- The horizontal tune is **36.44**
- Maximum orbit distortion in dipoles

$$y_0 = \frac{\sqrt{1.5 \cdot 1.5}}{2\sin(36.44\pi)} \cdot 10^{-3} \approx 1\text{mm}$$

- For quadrupole displacement with **1mm**, the distortion is $y_0 \approx 8\text{mm}$!!!
- Magnet alignment is critical



Many orbit errors' effect

- Consider random distribution of errors in N magnets
- The expectation value is given by

$$< u(s) >= \frac{\sqrt{\beta(s)}}{2\sqrt{(2)}\sin(\pi\nu)} \sum_i \beta_i \delta u_i' = \frac{\sqrt{\beta(s)\langle\beta\rangle}\sqrt{N}}{2\sqrt{(2)}\sin(\pi\nu)} \frac{(\delta B l)_{\rm rms}}{B\rho}$$

- Example:
  - In the SNS ring, there are **32** dipoles and **54** quadrupoles
  - The expectation value of the orbit distortion in the dipoles

$$y_0 = \frac{\sqrt{6 \cdot 6}\sqrt{32}}{2\sqrt{2}\sin(6.2\pi)} \cdot 10^{-3} \approx 2\text{cm}$$

  - And in the quadrupoles

$$y_0 = \frac{\sqrt{30 \cdot 30}\sqrt{54}}{2\sqrt{2}\sin(6.2\pi)} \cdot 10^{-3} \approx 13\text{cm}$$

Transport of orbit distortion due to dipole kick

- Consider a transport matrix between positions 1 and 2

$$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

- The transport of transverse coordinates is written as

$$\begin{aligned} u_2 &= m_{11}u_1 + m_{12}u'_1 \\ u'_2 &= m_{21}u_1 + m_{22}u'_1 \end{aligned}$$

- Consider a single dipole kick at position 1 $\delta u'_1 = \dfrac{\delta(Bl)}{B\rho}$
- Then, the first equation may be rewritten

$$u_2 + \delta u_2 = m_{11}u_1 + m_{12}(u'_1 + \delta u'_1) \to \delta u_2 = m_{12}\delta u'_1$$

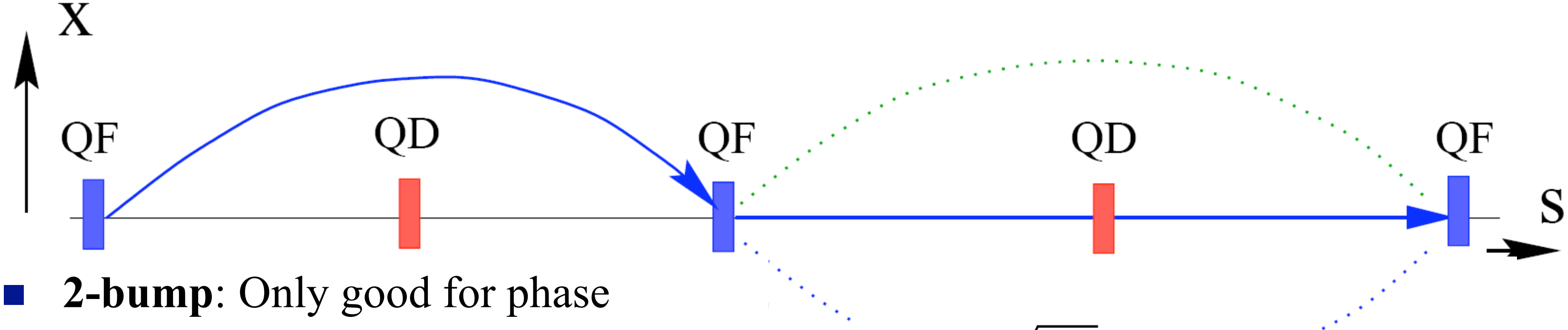
- Replacing the coefficient from the general betatron matrix

$$\delta u_2 = \sqrt{\beta_1\beta_2}\sin(\phi_{12})\delta u'_1$$

Correcting the orbit distortion

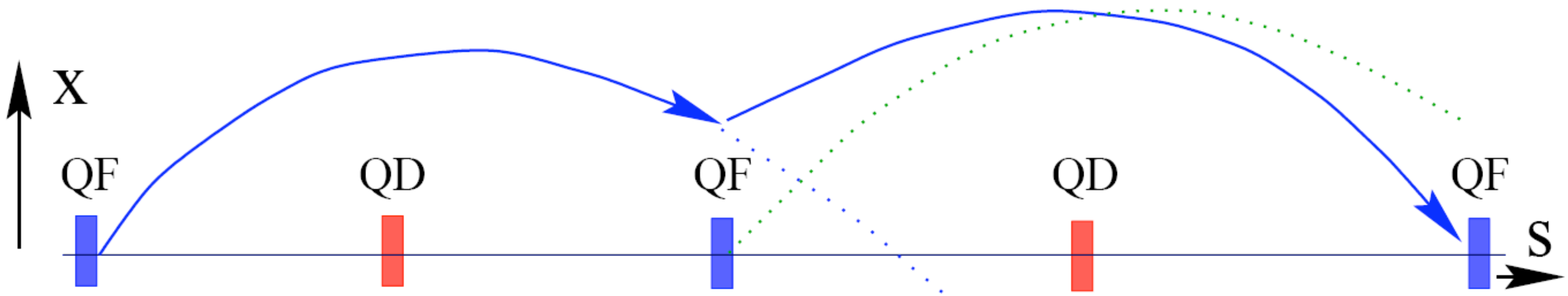
- Place horizontal and vertical dipole correctors close to the corresponding quads
- Simulate (random distribution of errors) or measure orbit in Beam position monitors (downstream of the correctors)
- Minimize orbit distortion with several methods
  - Globally
    - Harmonic , which minimizes components of the orbit frequency response after a Fourier analysis
    - Most efficient corrector (MICADO), finding the most efficient corrector for minimizing the rms orbit
    - Least square fitting
  - Locally
    - Sliding Bumps
    - Singular Value Decomposition (SVD)

Orbit bumps



- **2-bump**: Only good for phase advance equal π between correctors
- Sensitive to lattice and BPM errors
- Large number of correctors

$$\delta u_2' = \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \delta u_1'$$

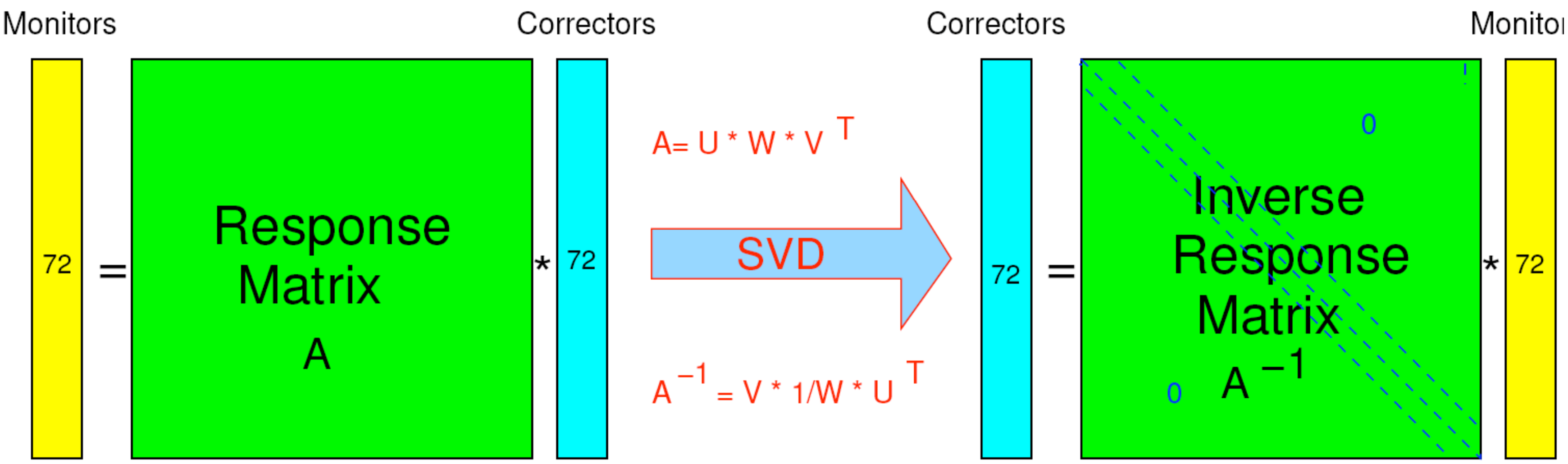


- **3-bump**: works for any lattice
- Need large number of correctors
- No control of angles

$$\frac{\sqrt{\beta_1}}{\sin\phi_{23}} \delta u_1' = \frac{\sqrt{\beta_2}}{\sin\phi_{31}} \delta u_2' = \frac{\sqrt{\beta_3}}{\sin\phi_{12}} \delta u_3'$$

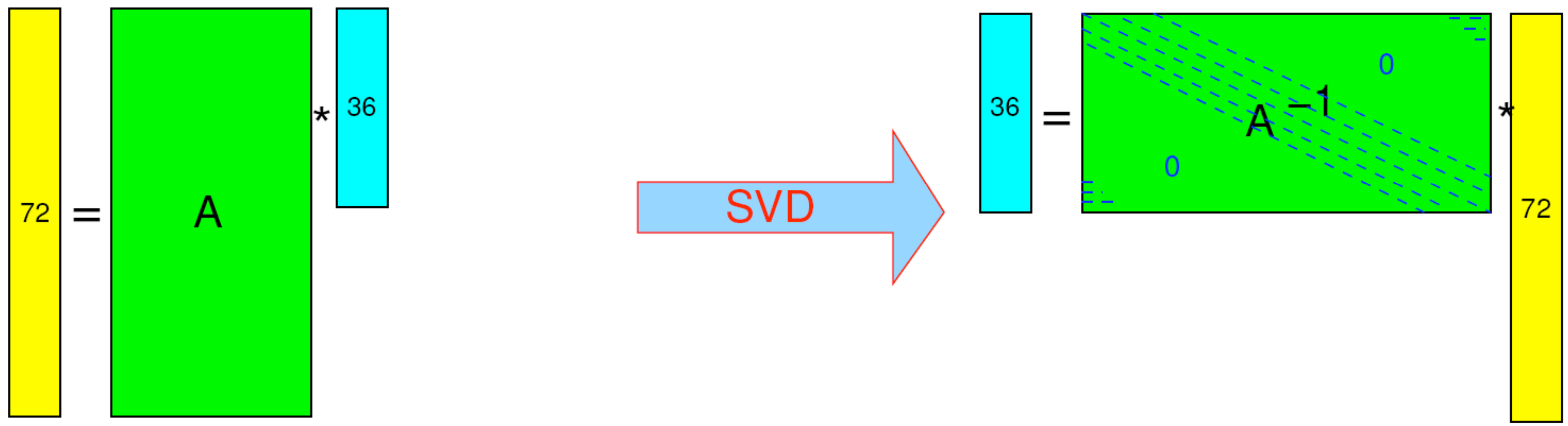
Singular Value Decomposition example

72 monitors / 72 correctors



=> Minimization of the RMS orbit (=0 in case of "Matrix Inversion" using all Eigenvalues)

72 monitors / 36 correctors



=> Minimization of the RMS orbit (monitor averaging)

M. Boege, CAS 2003

Orbit feedback

- Closed orbit stabilization performed using slow and fast orbit feedback system.
- Slow operates every a few seconds (~30s for ESRF storage ring) and uses complete set of BPMs (~200 at ESRF) for both planes
- Efficient in correcting distortion due to current decay in magnets or other slow processes
- Fast orbit correction system operates in a wide frequency range (.1Hz to 150Hz for the ESRF) correcting distortions induced by quadrupole and girder vibrations.
- Local feedback systems used to damp oscillations in areas where beam stabilization is critical (interaction points, insertion devices)

	β value at the BPM location	Rms motion without feedback	Rms motion with feedback	Rms motion / rms size
Horizontal	36 m	5 to 12 μm	1.2 to 2.2 μm	0.3 to 0.6 %
Vertical	5.6 m	1.5 to 2.5 μm	.8 to 1.2 μm	7 to 10 %

Gradient error and optics distortion

- Key issue for the performance -> super-periodicity preservation -> only structural resonances excited
- Broken super-periodicity -> excitations of all resonances
- Causes
  - Errors in quadrupole strengths (random and systematic)
  - Injection elements
  - Higher-order multi-pole magnets and errors
- Observables
  - Tune-shift
  - Beta-beating
  - Excitation of integer and half integer resonances

Gradient error

- Consider the transfer matrix for one turn

$$\mathcal{M}_0 = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ -\gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - \alpha_0 \sin(2\pi Q) \end{pmatrix}$$

- Consider a gradient error in a quad. In thin element approximation the quad matrix with and without error are

$$m_0 = \begin{pmatrix} 1 & 0 \\ -K_0(s)ds & 1 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 1 & 0 \\ -(K_0(s) + \delta K)ds & 1 \end{pmatrix}$$

- The new 1-turn matrix is $\mathcal{M} = m m_0^{-1} \mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ -\delta K ds & 1 \end{pmatrix} \mathcal{M}_0$ which yields

$$\mathcal{M}_0 = \begin{pmatrix} \cos(2\pi Q) + \alpha_0 \sin(2\pi Q) & \beta_0 \sin(2\pi Q) \\ \delta K ds(\cos(2\pi Q) - \alpha_0 \sin(2\pi Q)) - \gamma_0 \sin(2\pi Q) & \cos(2\pi Q) - (\delta K ds \beta_0 + \alpha_0) \sin(2\pi Q) \end{pmatrix}$$

Gradient error and tune-shift

- Consider a new matrix after 1 turn with a new tune $\chi = 2\pi(Q+\delta Q)$

$$\mathcal{M}^\star = \begin{pmatrix} \cos(\chi)+\alpha_0\sin(\chi) & \beta_0\sin(\chi) \\ -\gamma_0\sin(\chi) & \cos(\chi)-\alpha_0\sin(\chi) \end{pmatrix}$$

- The traces of the two matrices describing the 1-turn should be equal $\quad \mathrm{Tra}(\mathcal{M}^\star) = \mathrm{Tra}(\mathcal{M})$

  which gives $2\cos(2\pi Q) - \delta K ds \beta_0 \sin(2\pi Q) = 2\cos(2\pi(Q+\delta Q))$

- Developing the left hand side

$$\cos(2\pi(Q+\delta Q)) = \cos(2\pi Q)\underbrace{\cos(2\pi\delta Q)}_{1} - \sin(2\pi Q)\underbrace{\sin(2\pi\delta Q)}_{2\pi\delta Q}$$

  and finally $\quad 4\pi\delta Q = \delta K ds \beta_0$

- For a quadrupole of finite length, we have

$$\delta Q = \frac{1}{4\pi}\int_{s_0}^{s_0+l} \delta K \beta_0 ds$$

Gradient error and beta distortion

- Consider the unperturbed transfer matrix for one turn

$$\mathcal{M}_0 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = B \cdot A \;\; \text{with} \;\; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\; \text{and} \;\; \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

- Introduce a gradient perturbation between the two matrices

$$\mathcal{M}_0^\star = \begin{pmatrix} m_{11}^\star & m_{12}^\star \\ m_{21}^\star & m_{22}^\star \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\delta K ds & 1 \end{pmatrix} A$$

- Recall that $m_{12} = \beta_0 \sin(2\pi Q)$ and write the perturbed term as

$$m_{12}^\star = (\beta_0 + \delta\beta)\sin(2\pi(Q+\delta Q)) = \delta\beta \sin(2\pi Q) + 2\pi\delta Q \beta_0 \cos(2\pi Q)$$

- On the other hand

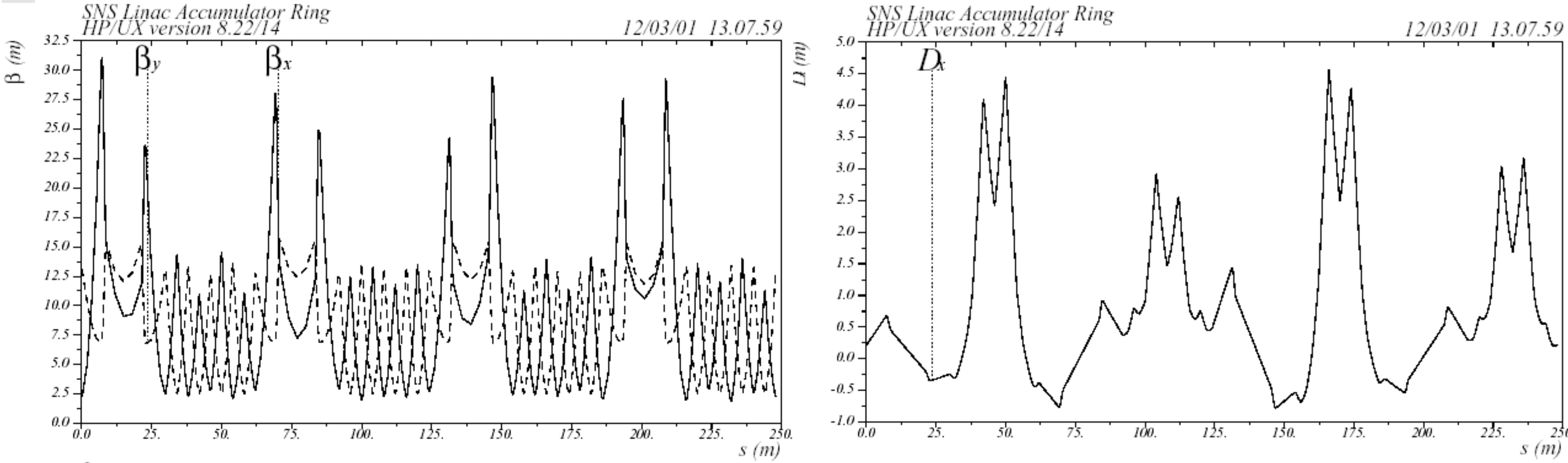
$$m_{12}^\star = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12}} - a_{12}b_{12}\delta K ds = \beta_0 \sin(2\pi Q) - a_{12}b_{12}\delta K ds$$

and $a_{12} = \sqrt{\beta_0 \beta(s_1)} \sin\psi, \; b_{12} = \sqrt{\beta_0\beta(s_1)} \sin(2\pi Q - \psi)$

- Equating the two terms and integrating through the quad

$$\frac{\delta\beta}{\beta_0} = -\frac{1}{2\sin(2\pi Q)} \int_{s_1}^{s_1+l} \beta(s)\delta K(s) \cos(2\psi - 2\pi Q) ds$$

Example: Gradient error in the SNS storage ring



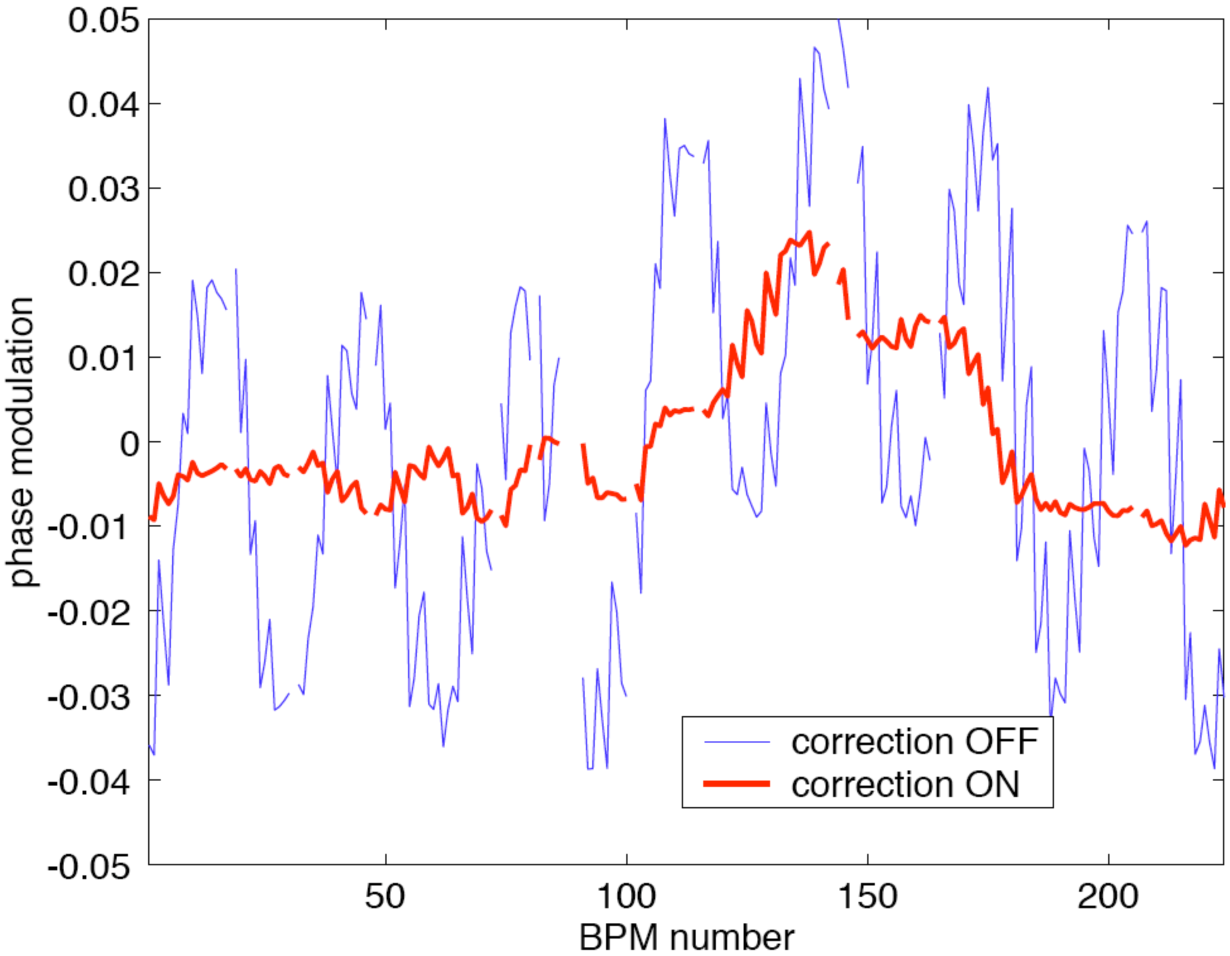
- Consider **18** focusing arc quads in the SNS ring with **1%** gradient error. In this location **β=12m**. The length of the quads is **0.5m**
- The tune-shift is $\delta Q = \frac{1}{4\pi} 18 \cdot 12 \frac{0.01}{5.6567} 0.5 = 0.015$
- For a random distribution of errors the beta beating is

$$\frac{\delta\beta}{\beta_0}_{\rm rms} = -\frac{1}{2\sqrt{2}|\sin(2\pi Q)|} (\sum_i \delta k_i^2 \beta_i^2)^{1/2}$$

- Optics functions beating > 20% by putting random errors (1% of the gradient) in high dispersion quads of the SNS ring
- Justifies the choice of TRIM windings strength

Example: Gradient error in the ESRF storage ring

- Consider **128** focusing arc quads in the SNS ring with **0.1%** gradient error. In this location <**β**>=**30m**. The length of the quads is around **1m**



- The tune-shift is

$$\delta Q = \frac{1}{4\pi} 128 \cdot 30 \frac{0.001}{20} 1 = 0.014$$

Random gradient error distribution

- For a random distribution of errors the beta variation (beating) is

$$\frac{\delta\beta}{\beta_0}_{\rm rms} = -\frac{1}{2\sqrt{2}|\sin(2\pi Q)|}\left(\sum_i \delta k_i^2 \beta_i^2\right)^{1/2}$$

- Optics functions beating >10% by putting random errors (0.1% of the gradient) in high dispersion quads of the ESRF storage ring
- Justifies the choice of quadrupole corrector strength

Gradient error correction

- Windings on the core of the quadrupoles or individual correction magnets (TRIM)
- Simulation by introducing random distribution of quadrupole errors
- Compute tune-shift and optics function beta distortion
- Move working point close to integer and half integer resonance
- Minimize beta wave or quadrupole resonance width with TRIM windings
- To correct certain resonance harmonics N, strings should be powered accordingly
- Individual powering of TRIM windings can provide flexibility and beam based alignment of BPM

Linear coupling

CERN

- Betatron motion is coupled in the presence of skew quadrupoles
- The field is $(B_x, B_y) = k(x, y)$ and Hill's equations are coupled
- Motion still linear with two new eigen-mode tunes, which are always split. In the case of a thin quad:

$$\delta Q \propto |k|\sqrt{\beta_x \beta_y}$$

- Coupling coefficients

$$|C_\pm| = \left|\frac{1}{2\pi}\oint ds k(s)\sqrt{\beta_x(s)\beta_y(s)} e^{i(\phi_x \pm \phi_y - (Q_x \pm Q_y - q_\pm)2\pi s/C)}\right|$$

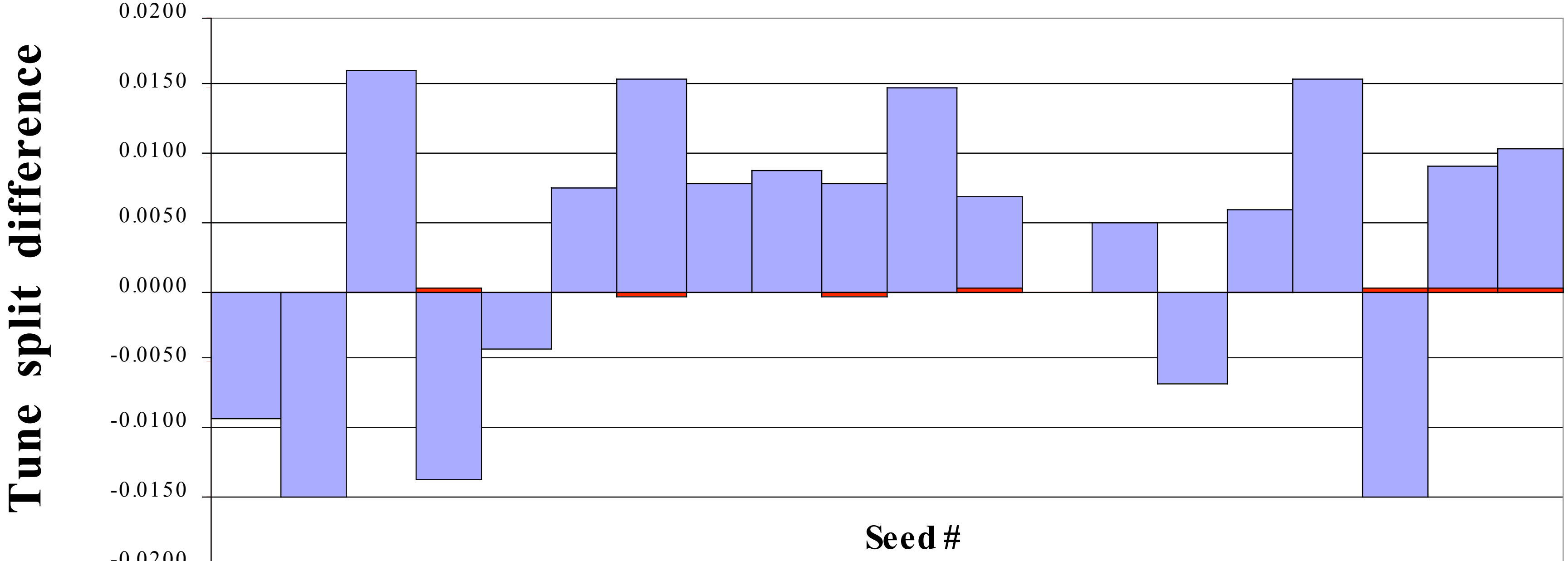
- As motion is coupled, vertical dispersion and optics function distortion appears
- Causes:
  - Random rolls in quadrupoles
  - Skew quadrupole errors
  - Off-sets in sextupoles

Linear coupling correction

- Introduce skew quadrupole correctors
- Simulation by introducing random distribution of quadrupole errors
- Correct globally/locally coupling coefficient (or resonance driving term)
- Correct optics distortion (especially vertical dispersion)
- Move working point close to coupling resonances and repeat
- Correction especially critical for flat beams

Example: Coupling correction for the SNS ring

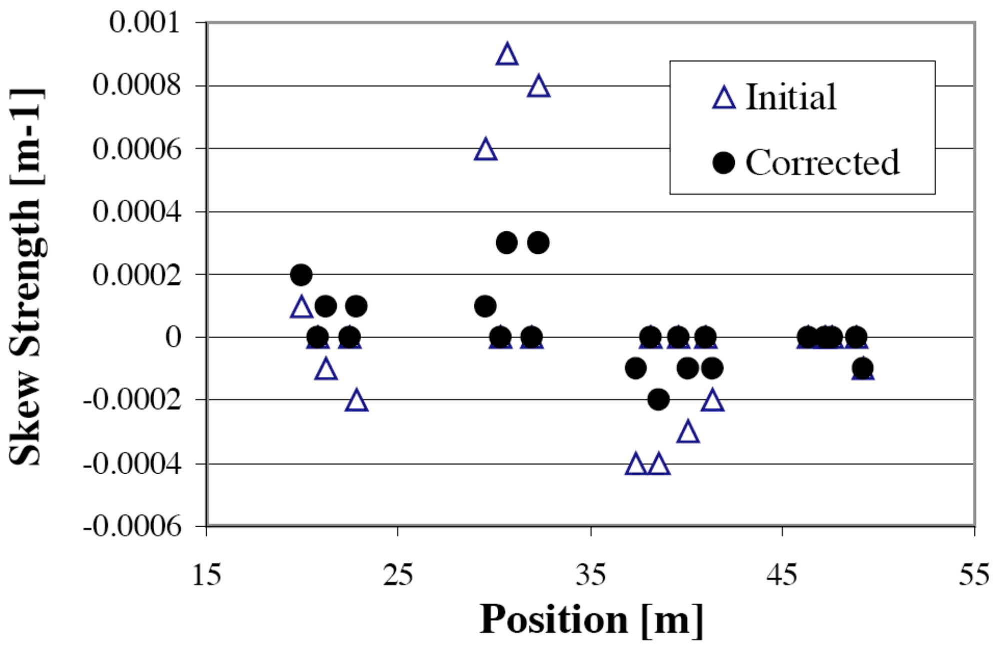
- Local decoupling by super period using 16 skew quadrupole correctors
- Results of Q_x=6.23 Q_y=6.20 after a 2 mrad quad roll
- Additional 8 correctors used to compensate vertical dispersion

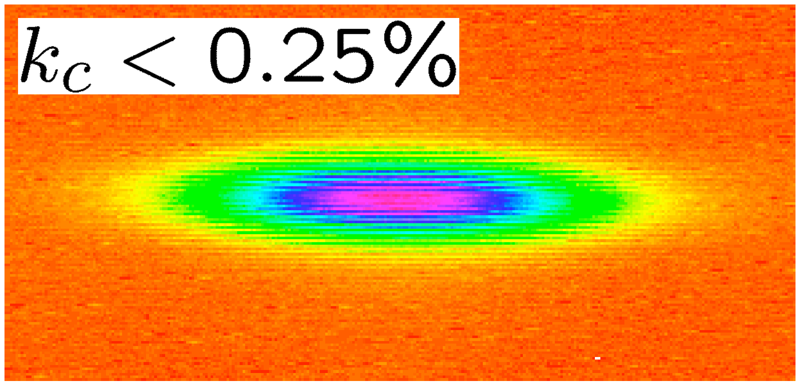


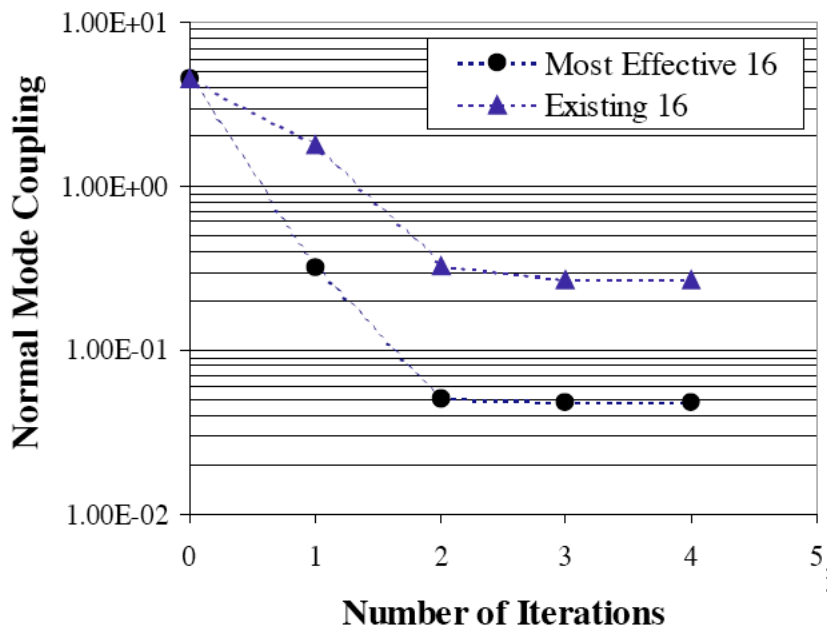
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Before Correction	-0.009	-0.014	0.016	-0.013	-0.004	0.007	0.015	0.008	0.008	0.007	0.014	0.006	0.000	0.005	-0.006	0.006	0.015	-0.015	0.009	0.010
After correction	0.000	0.000	0.000	0.000	-0.000	0.000	-0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	-0.000	-0.000	0.000	-0.000	0.000	0.000	0.000

Example: Coupling correction for the ESRF ring

- Local decoupling using 16 skew quadrupole correctors and coupled response matrix reconstruction
- Achieved correction of below 0.25% reaching vertical emittance of below 10pm







R. Nagaoka, EPAC 2000

Chromaticity

- Linear equations of motion depend on the energy (term proportional to dispersion)
- Chromaticity is defined as: $\xi_{x,y} = -\frac{\delta Q_{x,y}}{\delta P/P}$
- Recall that the gradient is $K = \frac{G}{B\rho} = \frac{eG}{P} \rightarrow \frac{\delta K}{K} = \mp\frac{\delta P}{P}$
- This leads to dependence of tunes and optics function on energy
- For a linear lattice the tune shift is:

$$\delta Q_{x,y} = \frac{1}{4\pi}\oint \beta_{x,y}\delta K(s)ds = \frac{1}{4\pi}\frac{\delta P}{P}\oint \beta_{x,y}K(s)ds$$

- So the natural chromaticity is:

$$\xi_{x,y} = -\frac{1}{4\pi}\oint \beta_{x,y}K(s)ds$$

Example: Chromaticity in the SNS ring

- In the SNS ring, the natural chromaticity is –7.
- Consider that momentum spread $\frac{\delta P}{P} = \pm 1$
- The tune-shift for off-momentum particles is

$$\delta Q_{x,y} = \xi_{x,y} \frac{\delta P}{P} = \pm 0.07$$

- In order to correct chromaticity introduce particles which can focus off-momentum particle

↓

Sextupoles

Chromaticity from sextupoles

- The sextupole field component in the x-plane is: $B_y = \frac{S}{2} x^2$
- In an area with non-zero dispersion $x = x_0 + D\frac{\delta P}{P}$
- Than the field is

$$B_y = \frac{S}{2}x_0^2 + \underbrace{SD\frac{\delta P}{P}x_0}_{\text{quadrupole}} + \underbrace{\frac{S}{2}D^2\frac{\delta P}{P}^2}_{\text{dipole}}$$

- Sextupoles introduce an equivalent focusing correction

$$\delta K = SD\frac{\delta P}{P}$$

- The sextupole induced chromaticity is

$$\xi^S_{x,y} = -\frac{1}{4\pi}\oint \beta_{x,y}(s)S(s)D_x(s)ds$$

- The total chromaticity is the sum of the natural and sextupole induced chromaticity

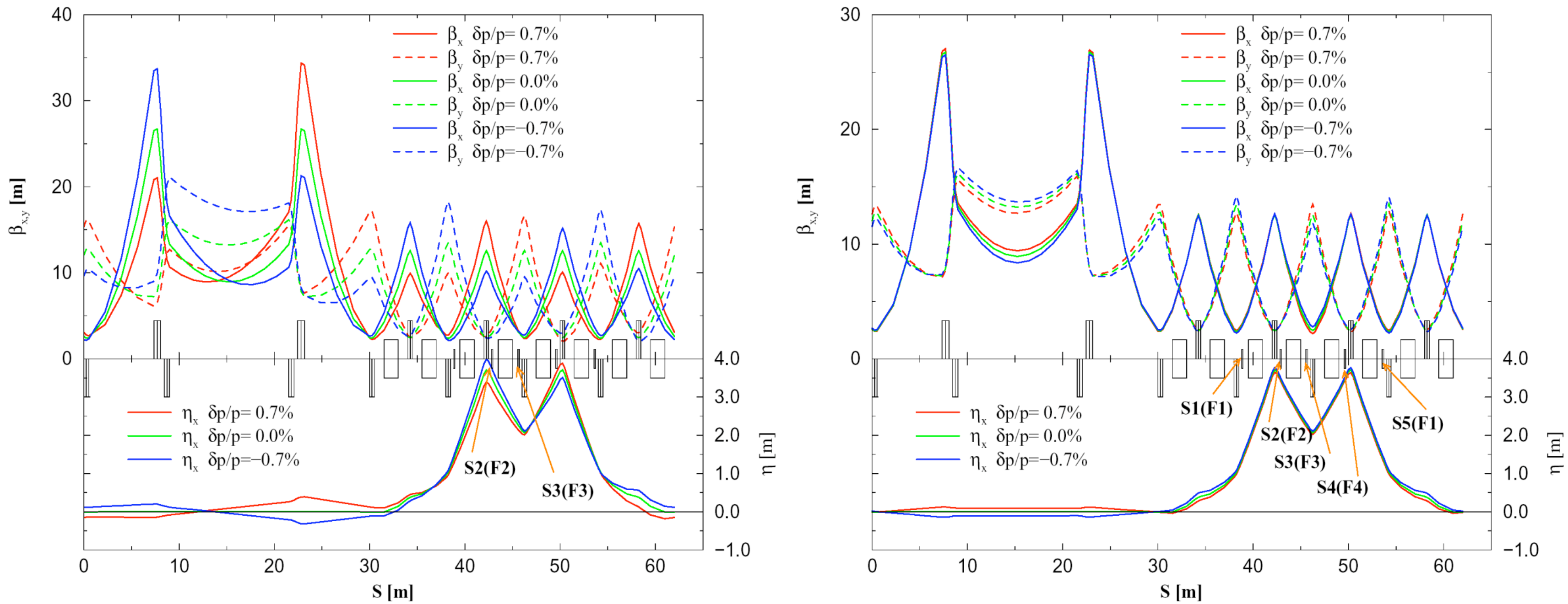
$$\xi^{tot}_{x,y} = -\frac{1}{4\pi}\oint \beta_{x,y}(s)(S(s)D_x(s) + k(s))ds$$

Chromaticity correction

- Introduce sextupoles in high-dispersion areas
- Tune them to achieve desired chromaticity
- Two families are able to control horizontal and vertical chromaticity
- Sextupoles introduce non-linear fields (chaotic motion)
- Sextupoles introduce tune-shift with amplitude
- Example:
  - The SNS ring has natural chromaticity of **–7**
  - Placing two sextupoles of length **0.3m** in locations where **β=12m**, and the dispersion **D=4m**
  - For getting **0** chromaticity, their strength should be

$$S = \frac{7 \cdot 4\pi}{12 \cdot 4 \cdot 2 \cdot 0.3} \approx 3\mathrm{m}^{-3}$$ or a gradient of **17.3 T/m²**

Two vs. four families for chromaticity correction



- Two families of sextupoles not enough for correcting off-momentum optics functions' distortion and second order chromaticity
- Solutions:
  - Place sextupoles accordingly to eliminate second order effects (difficult)
  - Use more families (4 in the case of of the SNS ring)
- Large optics function distortion for momentum spreads of **±0.7%**,when using only two families of sextupoles
- Absolute correction of optics beating with four families

Chromaticity correction

- Introduce sextupoles in high-dispersion areas (not easy to find)
- Tune them to achieve desired chromaticity
- Two families are able to control horizontal and vertical chromaticity
- Sextupoles introduce non-linear fields (chaotic motion)
- Sextupoles introduce tune-shift with amplitude
- Example:
  - The ESRF ring has natural chromaticity of **-130**
  - Placing 32 sextupoles of length **0.4m** in locations where **β=30m**, and the dispersion **D=0.3m**
  - For getting **0** chromaticity, their strength should be

$$S = \frac{130 \cdot 4\pi}{30 \cdot 0.3 \cdot 32 \cdot 0.4} \approx 14\mathrm{m}^{-3}$$

or a gradient of **280 T/m²**

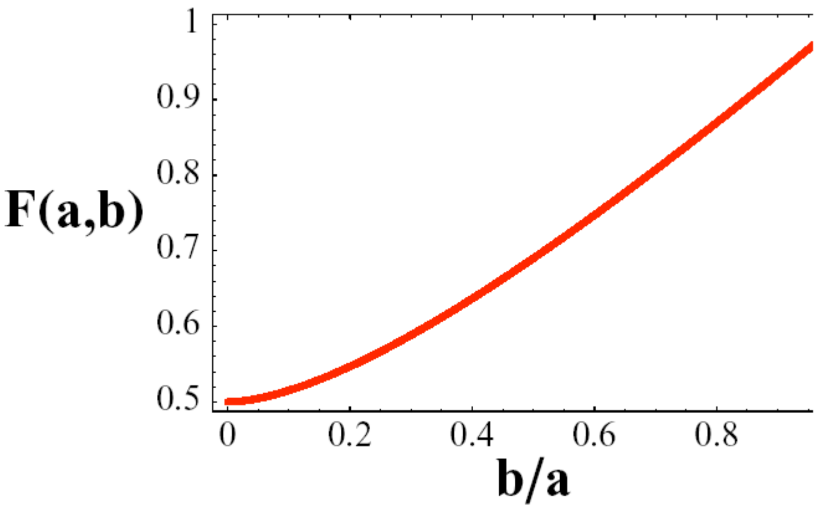
Eddy current sextupole component

$$\xi^{\text{eddy}}_{x,y}(t) = \pm\frac{1}{4\pi}\oint S^{\text{eddy}}(s,t)\eta_x(s)\beta_{x,y}(s)ds$$

Sextupole component due to Eddy currents in an elliptic vacuum chamber of a pulsing dipole

$$S^{\text{eddy}}(t) = \frac{1}{B\rho}\frac{d^2B_y}{dx^2} = \frac{1}{B\rho}\frac{\mu_0\sigma_c t\dot{B}_y}{h}F(a,b)$$

with $F(a,b) = \int_0^{\pi/2}\sin\phi\sqrt{\cos^2\phi + (b/a)^2\sin^2\phi}\,d\phi = 1/2\left[1 + \frac{b^2\text{arcsinh}(\sqrt{a^2-b^2}/b)}{a\sqrt{a^2-b^2}}\right]$



Taking into account

$$B_y(t) = \frac{B_{\text{max}}}{1+a_E}(a_E - \cos(\omega t))$$

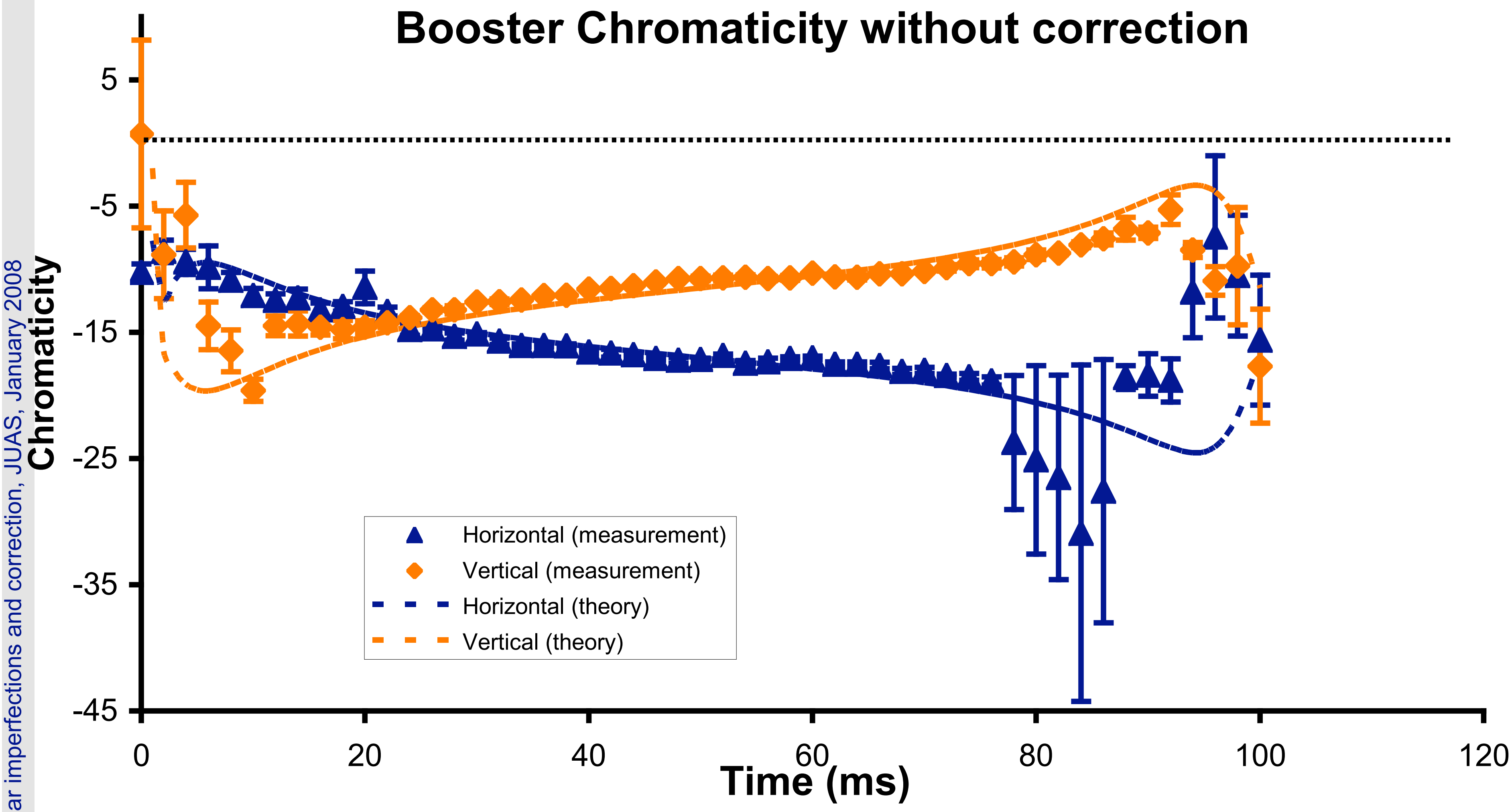
with

$$a_E = \frac{E_{\text{max}} + E_{\text{min}}}{E_{\text{max}} - E_{\text{min}}}$$

we get $S^{\text{eddy}}(t) = \frac{\mu_0\ \sigma_c\ t\ \omega}{h\ \rho}\frac{\sin(\omega t)}{a_E - \cos(\omega t)}F(a,b)$

ESRF booster example

Example: ESRF booster chromaticity



5.1) A **1GeV proton** ring with a **circumference** of **248m** has **18 1m-long** focusing quads with **gradient** of **5T/m**, with an horizontal and vertical **beta** function of **12m** and **2m** respectively. The **average beta** function around the ring is **8m**. With a **horizontal tune** of **6.23** and a vertical of **6.2**, compute the average orbit distortions on the quads given by **horizontal** and by **vertical** misalignments of **1mm**.What happens to the orbit distortions if the horizontal tune drops to **6.1** and **6.01**?

5.2) Three correctors are placed at locations with phase advance of **$\pi/4$** between them and beta functions of **12**, **2** and **12m**. How are the corrector kicks related to each other in order to achieve a closed 3-bump.

5.3) Consider a **400GeV** proton synchrotron with **108 3.22m-long** focusing and defocusing quads of **19.4 T/m**, with a horizontal and vertical **beta** of **108m** and **18m** in the focusing quads which is inversed for the defocusing ones. Find the tune change for systematic gradient errors of **1%** in the focusing and **0.5%** in the defocusing quads. What is the chromaticity of the machine?

5.4) Derive an expression for the resulting magnetic field when a normal sextupole with field $\mathbf{B = S/2\ x^2}$ is displaced by $\boldsymbol{\delta x}$ from its center position. At what type of fields correspond the resulting components? Do the same for an octupole with field $\mathbf{B = O/3\ x^3}$. What is the leading order multi-pole field error when displacing a general **2n**-pole magnet?