Non-linear dynamics

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Summary

- Driven oscillators and resonance condition
- Field imperfections and normalized field errors
- Perturbation treatment for a sextupole
- Poincaré section
- Chaotic motion
- Octupole effect and fringe fields
- Singe-particle diffusion
  - Dynamics aperture
  - Frequency maps

Damped and driven harmonic oscillator

- Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = 0$$

  - Q is the damping coefficient (amplitude decreases with time)
  - ω_0 is the Eigenfrequency of the harmonic oscillator

- An external force can pump energy into the system

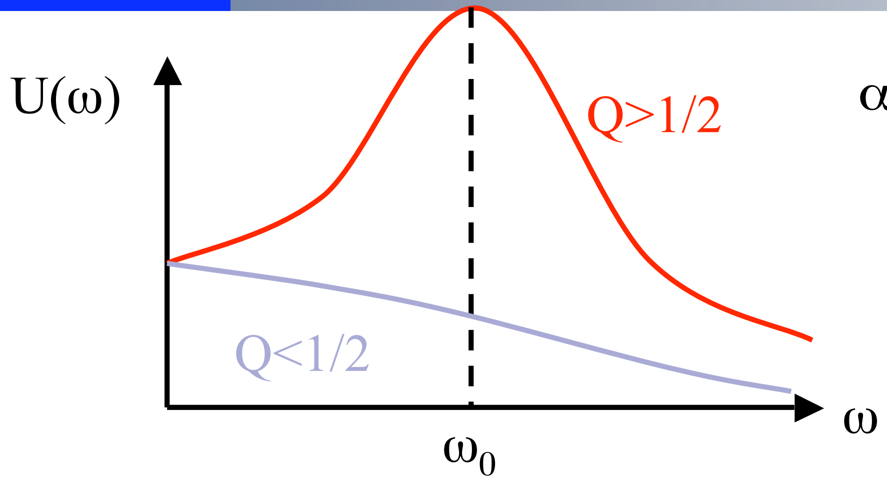
$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

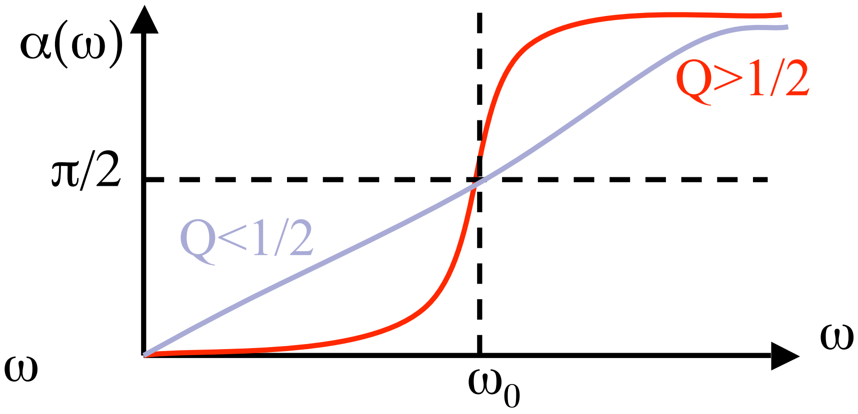
- General solution $u(t) = u_h(t) + u_{st}(t)$ with

$$u_{st}(t) = U(\omega)\cos(\omega t - a(\omega))$$

  - **ω** the frequency of the driven oscillation
  - Amplitude **U(ω)** can become large for certain frequencies

Resonance effect



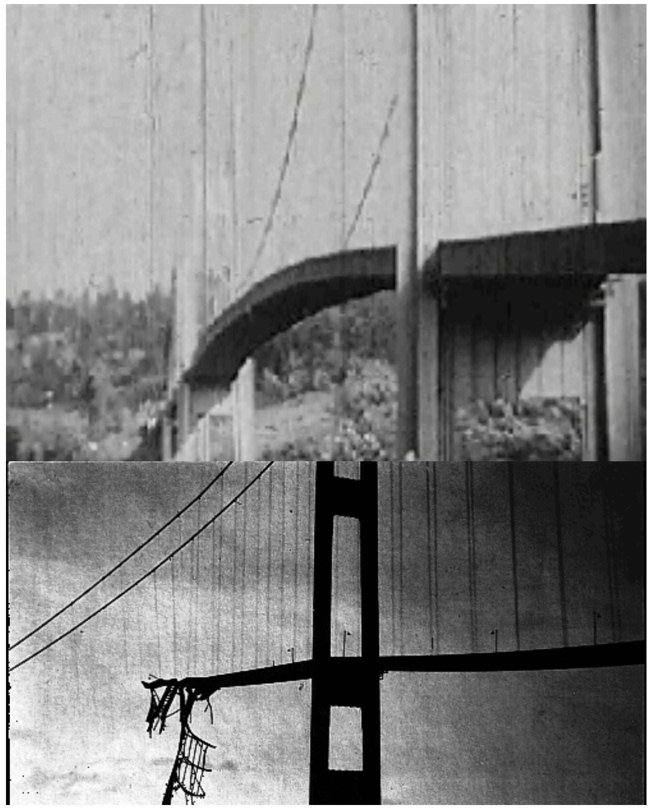


$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

excitation by strong wind on the eigenfrequencies



Perturbation in Hill's equations

- Hill's equations with driven harmonic force

$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = F(u(s), s)$$

where the F is the Lorentz force from perturbing fields

  - **Linear magnet imperfections**: derivation from the design dipole and quadrupole fields due to powering and alignment errors
  - **Time varying fields**: feedback systems (damper) and wake fields due to collective effects (wall currents)
  - **Non-linear magnets**: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
  - **Beam-beam interactions**: strongly non-linear field
  - **Space charge effects**: very important for high intensity beams
  - **non-linear magnetic field imperfections**: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

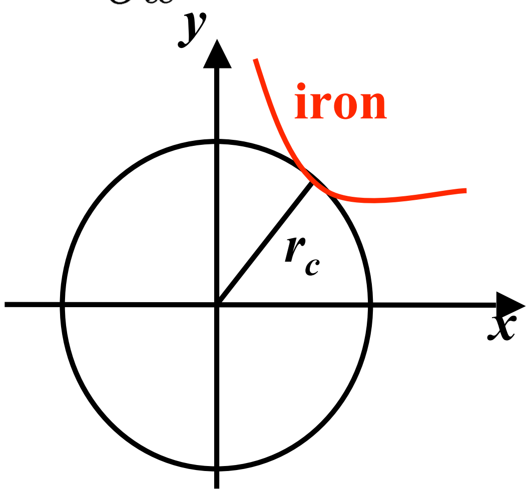
- Assuming a 2D field in x and y, the vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe)

$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}\ , \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



There exist a complex potential of $z = x+iy$ with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \,, \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \,, \; a_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4} B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note**: *n'=n-1* is the US convention

Localized Perturbation

- Periodic delta function

$$\delta_L(s-s_0)=\begin{cases}1 & \text{for 's' = } s_0\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \oint \delta_L(s-s_0)ds = 1$$

- Equation of motion for a single perturbation in the storage ring

$$\frac{d^2u(s)}{ds^2}+\omega_0^2u(s)=\delta_L(s-s_0)lF(u(s),s)$$

- Expanding in Fourier series the delta function

$$\frac{d^2u(s)}{ds^2}+\omega_0^2u(s)=\frac{l}{C}\sum_m \cos(2\pi m\frac{s}{C})F(u(s),s)$$

- Infinite number of driving frequencies!!!
- Recall that the driving force can be the result of any multi-pole

$$B_y+iB_x=\sum_{n=1}^{\infty}(b_n-ia_n)(x+iy)^{n-1}$$

Resonance conditions and tune diagram

- Equations of motion (u = x or y) including all multi-pole errors

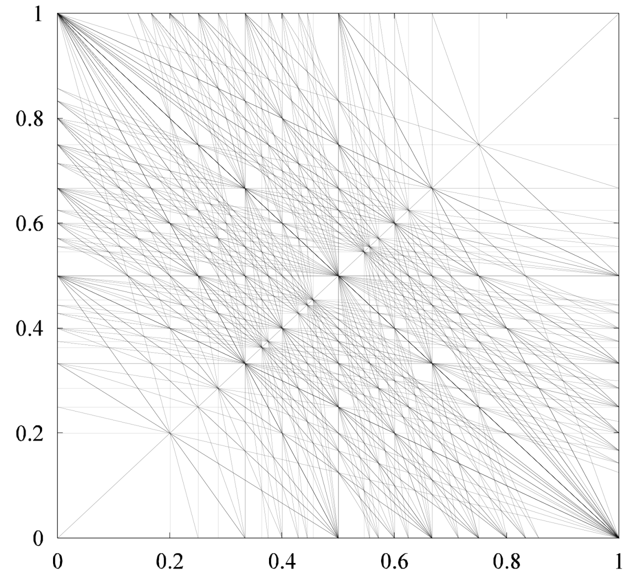
$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = \epsilon \sum_{n_x+n_y<n,m} a_{n_x,n_y,m} x^{n_x} y^{n_y} \cos(2\pi m \frac{s}{C})$$

- Solved with perturbation theory approach

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$$ with

$$u_0(s) = u_0 \cos(2\pi Q s/C + \phi_0) \text{ and } \omega_0 = \frac{2\pi}{C} Q$$

- At first order $\frac{d^2u_1(s)}{ds^2} + \omega_0^2 u_1(s) = \epsilon \sum_{n'_x<n_x, n'_y<n_y} \bar{a}_{n'_x,n'_y,m} \cos(\frac{2\pi s}{C}(n'_x Q_x + n'_y Q_y + m))$



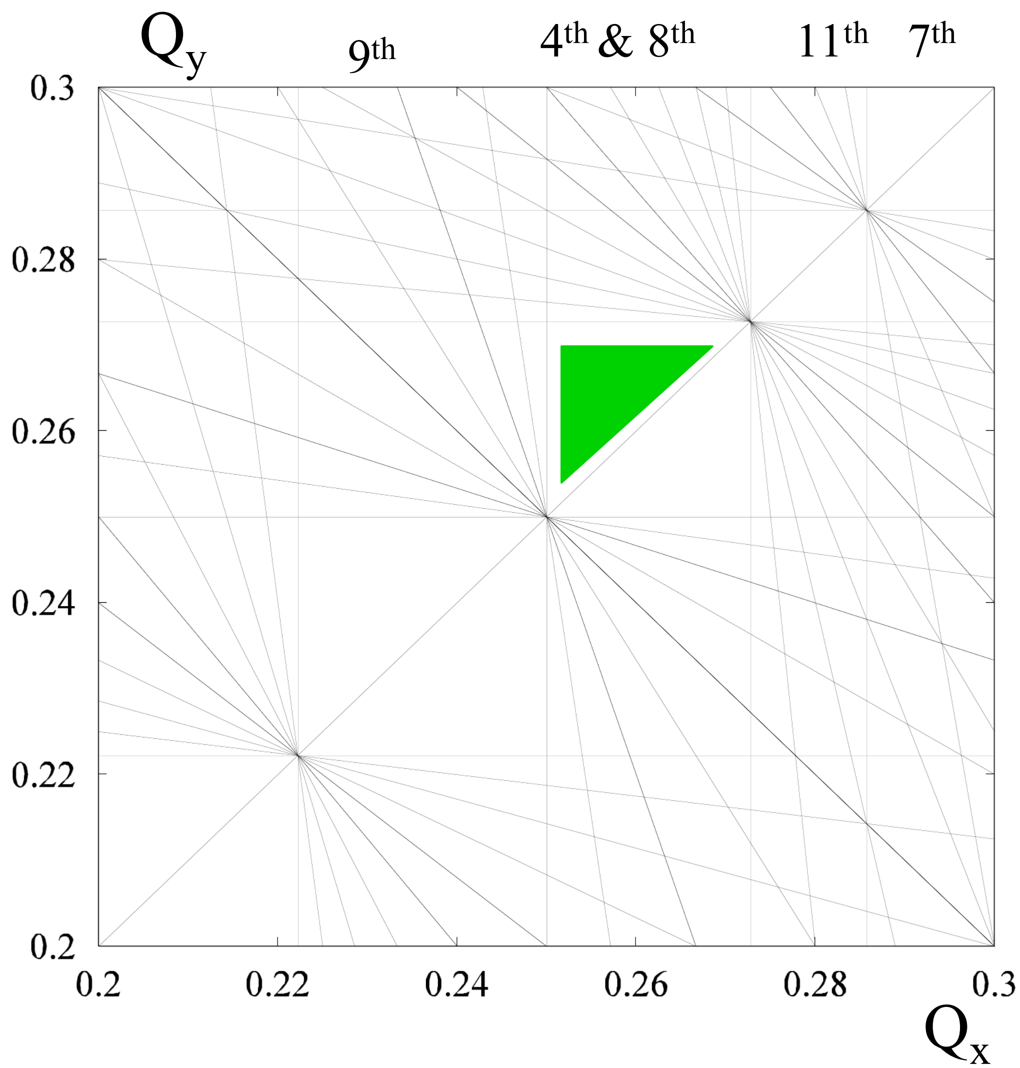
Resonance condition

$$n'_x Q_x + n'_y Q_y + m = 0$$

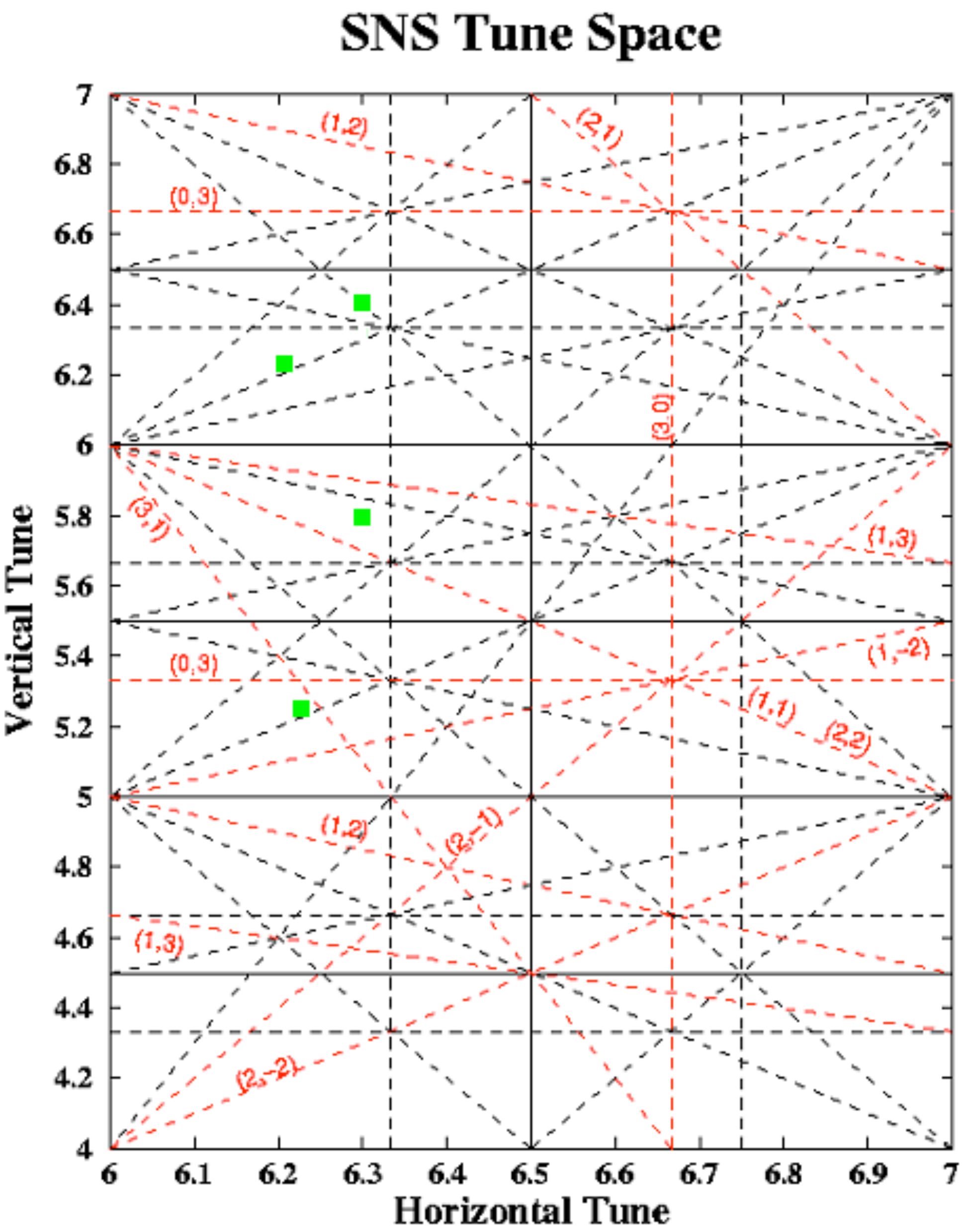
There are resonance lines everywhere !!!

Choice of the working point

- Regions with few resonances:
$$n'_x Q_x + n'_y Q_y + m = 0$$
- Avoid low order resonances
- < 12^{th} order for a proton beam without damping
- < 3^{rd} ⇔ 5^{th} order for electron beams with damping
- Close to coupling resonances: regions without low order resonances but relatively small!



Example: SNS Ring Tune Space



Tunability: 1 unit in horizontal, 3 units in vertical (2 units due to bump/chicane perturbation)

– **Structural resonances (up to 4th order)**

– **All other resonances (up to 3rd order)**

- **Working points considered**
  - **(6.30,5.80) - Old**
  - **(6.23,5.24)**
  - **(6.23,6.20) - Nominal**
  - **(6.40,6.30) - Alternative**

Single Sextupole Perturbation

- Consider a thin sextupole perturbation $$F(S) = \delta(s - s_0) l \frac{S}{2} x_0^2$$
- Equations of motion

$$\frac{d^2 x_1(s)}{ds^2} + \omega_0^2 x_1(s) = x_0^2 \frac{Sl}{2C} \sum_m \cos(2\pi m \frac{s}{C})$$

- With $$x_0(s) = x_0 \cos(\omega_0 s + \phi_0)$$
- The equation is written

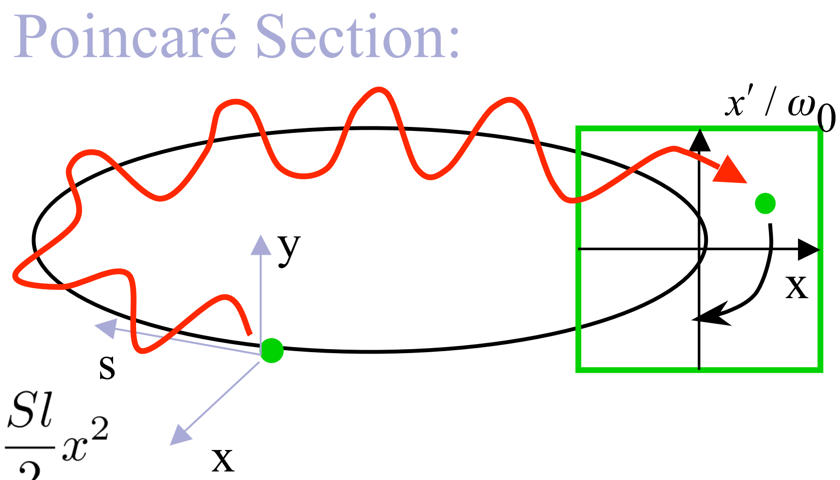
$$\frac{d^2 x_1(s)}{ds^2} + \omega_0^2 x_1(s) = \frac{Sl}{2C} A^2 \sum_m \cos(2\pi m \frac{s}{C}) + \frac{Sl}{8C} A^2 \sum_m \cos(2\pi (m \pm 2Q_x) \frac{s}{C})$$

- Resonance conditions $$\underbrace{Q_x + m = 0}_{\text{integer}} \qquad \underbrace{3Q_x + m = 0}_{\text{third integer}}$$
- No exact solution
- Need numerical tools to integrate equations of motion

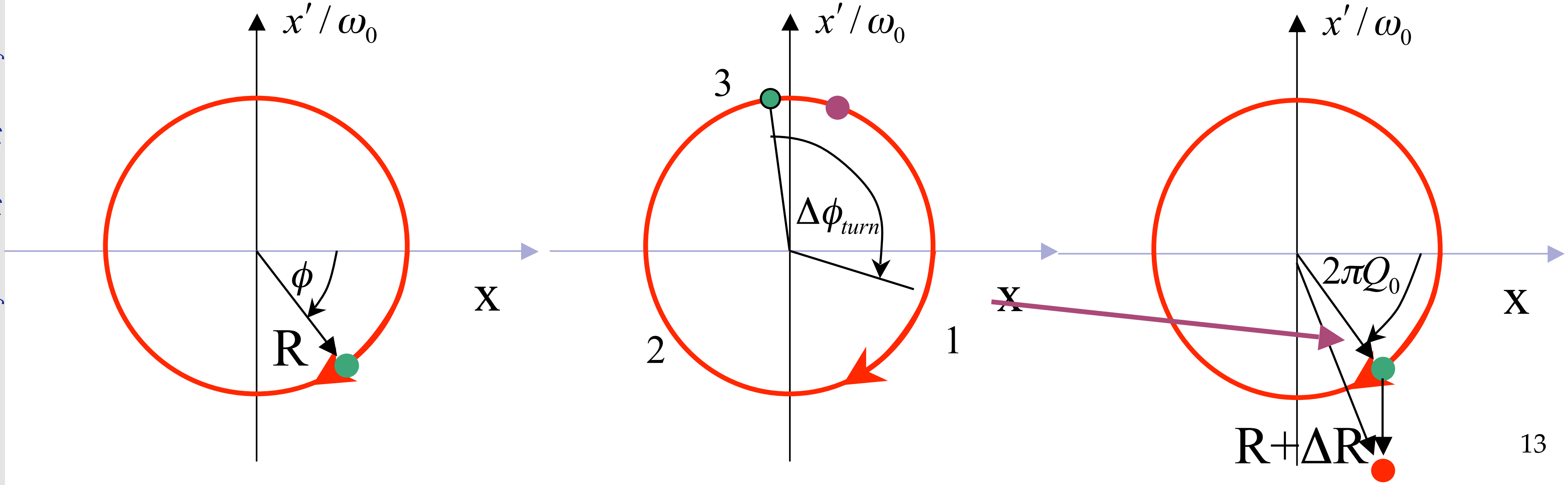
Poincaré Section

- Record the particle coordinates at one location (BPM)
- Unperturbed motion lies on a circle (simple rotation)
- Resonance is described by fixed points
- For a sextupole

$$\Delta x' = \oint F(s)ds \rightarrow \Delta x' = \frac{Sl}{2}x^2$$

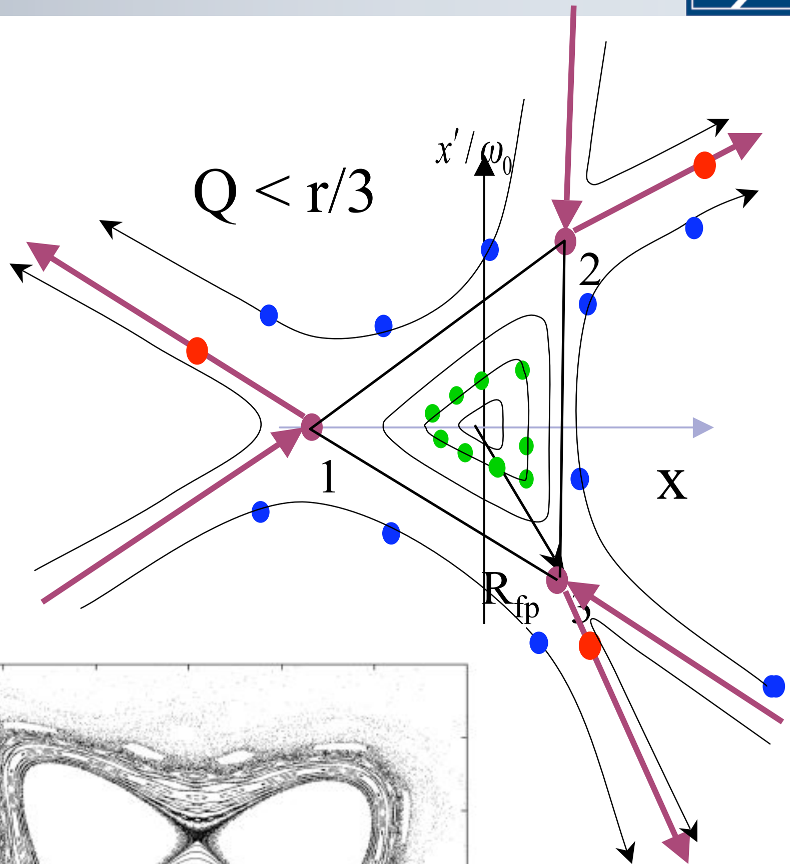


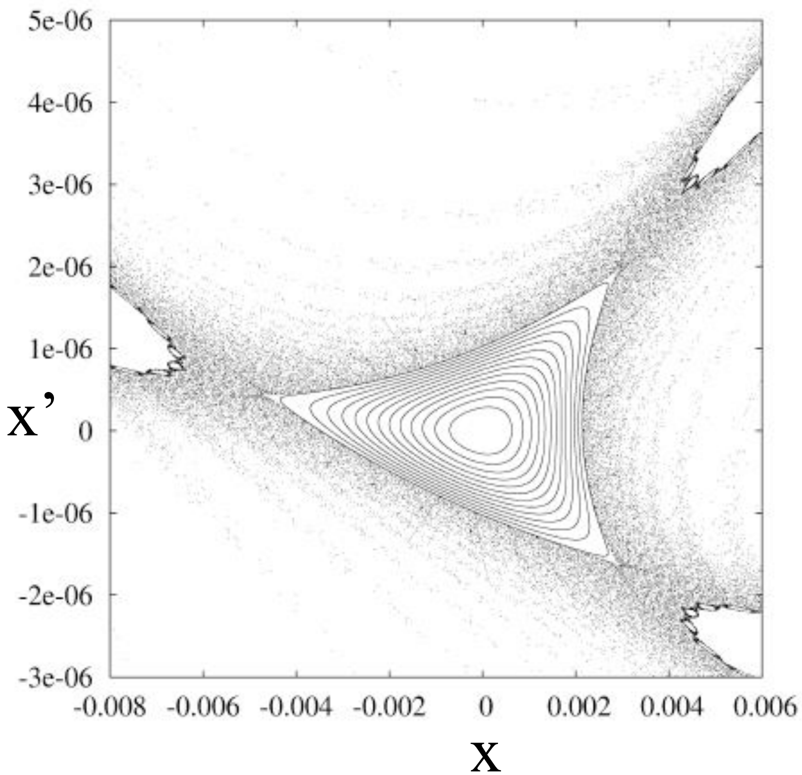
- The particle does not lie on a circle!
- The change of tune per turn is $\Delta Q \propto x^2$

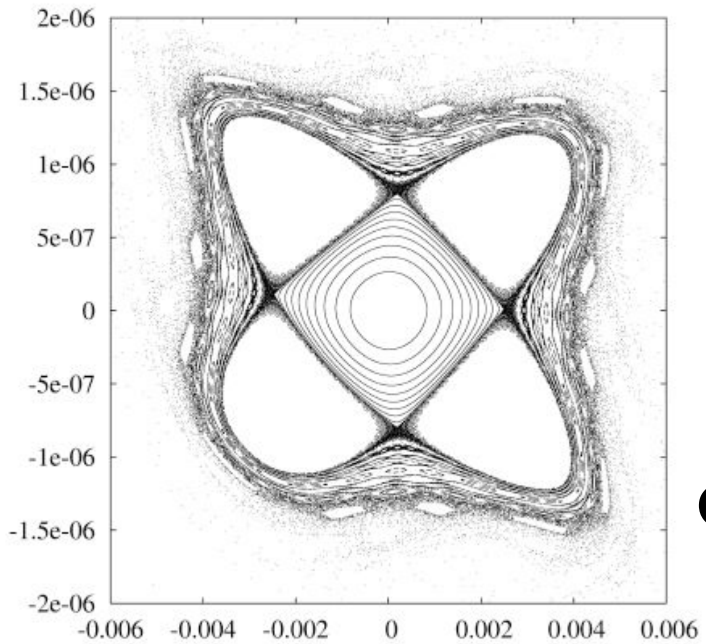


Topology of a sextupole resonance

- Small amplitude, regular motion
- Large amplitude, instability, chaotic motion and particle loss
- Separatrix: barrier between stable and unstable motion (location of unstable fixed points)







Octupole

Sextupole effects

9 first order terms:

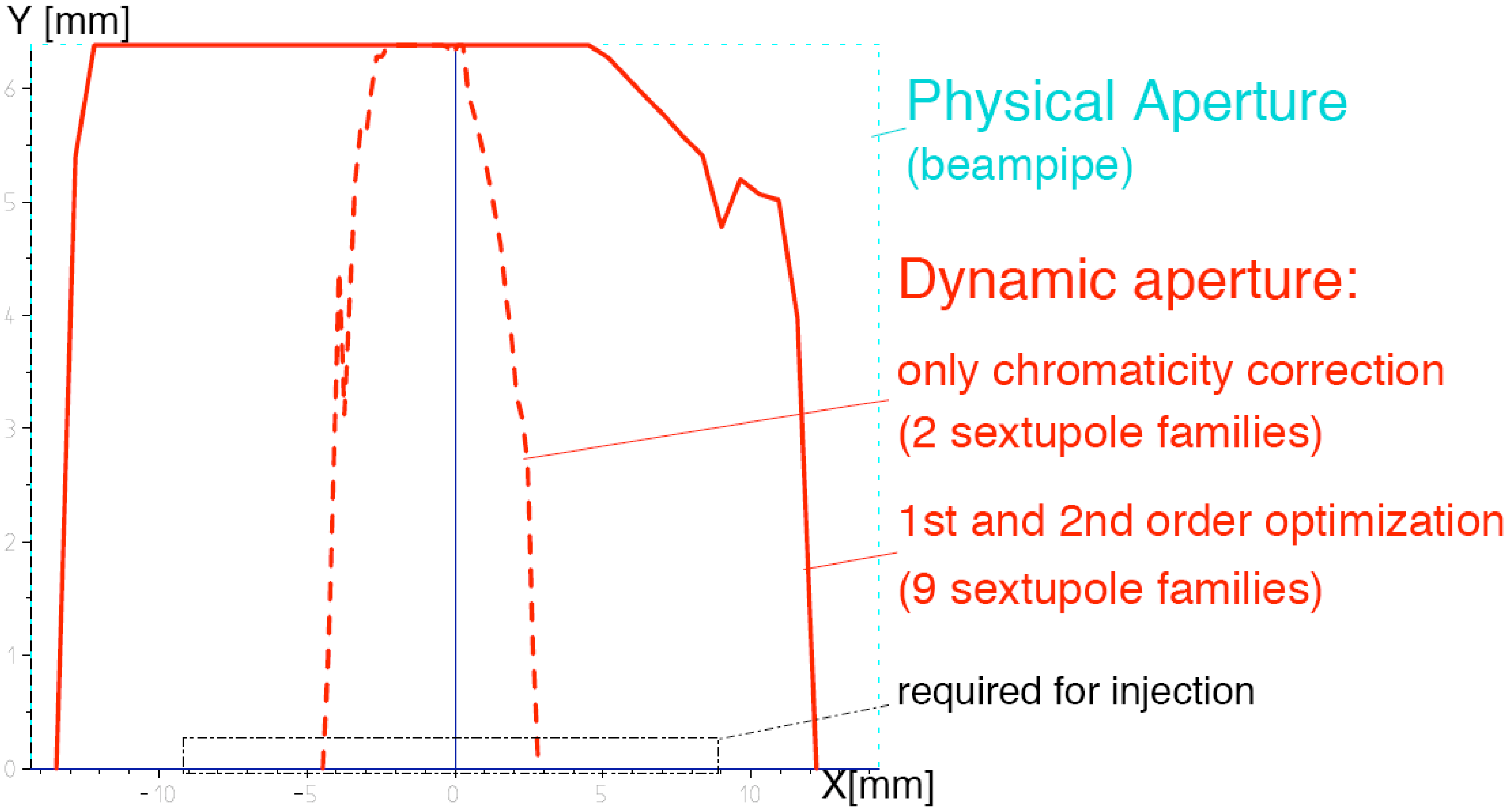
- 2 chromaticities ξ_x, ξ_y
- 2 off-momentum resonances $2Q_x, 2Q_y \to d\beta/d\delta \to \xi^{(2)} = \partial^2 Q/\partial\delta^2$
- 2 terms $\to$ integer resonances Q_x
- 1 term $\to$ 3^{rd} integer resonances $3Q_x$
- 2 terms $\to$ coupling resonances $Q_x \pm 2Q_y$

13 second order terms:

- 3 tune shifts with amplitude: $\partial Q_x/\partial J_x, \ \ \partial Q_x/\partial J_y = \partial Q_y/\partial J_x, \ \ \partial Q_y/\partial J_y$
- 8 terms $\to$ octupole like resonances: $4Q_x, 2Q_x \pm 2Q_y, 4Q_y, 2Q_x, 2Q_y$
- 2 second order chromaticities: $\partial^2 Q_x/\partial\delta^2$ and $\partial^2 Q_y/\partial\delta^2$

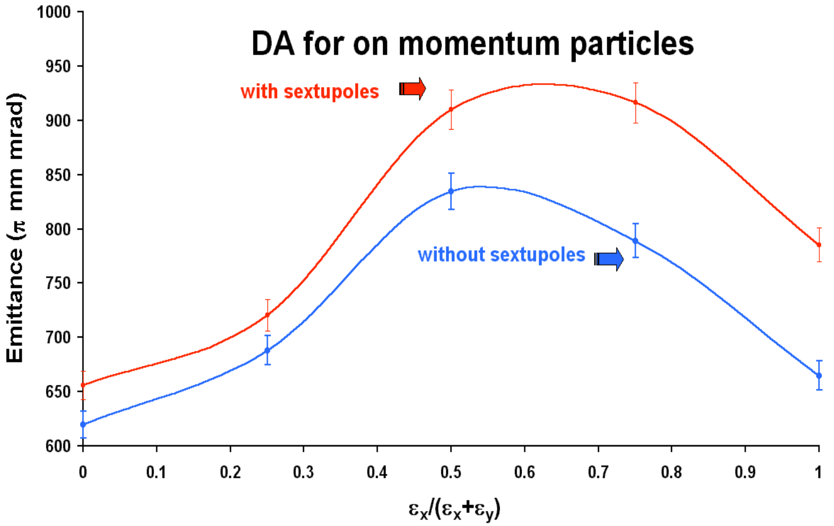
Optimization of Dynamic aperture

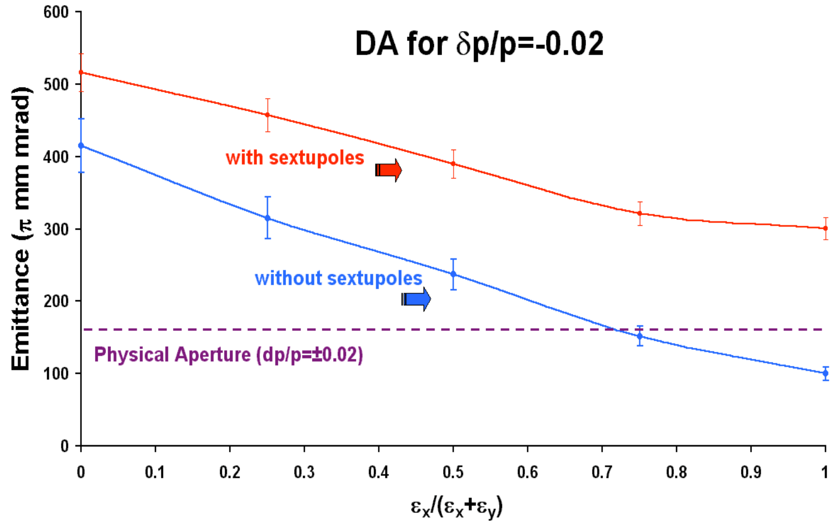
- Keep chromaticity sextupole strength low
- Include sextupoles in quadrupoles for more flexibility
- Try an interleaved sextupole scheme (-I transformer)
- Choose working point far from systematic resonances
- Iterate between linear and non-linear lattice



Dynamic Aperture for the SNS ring

Dynamic aperture tracking for on momentum particles (left) and for dp/p = -0.02 (right), without (blue) and with (red) chromatic sextupoles

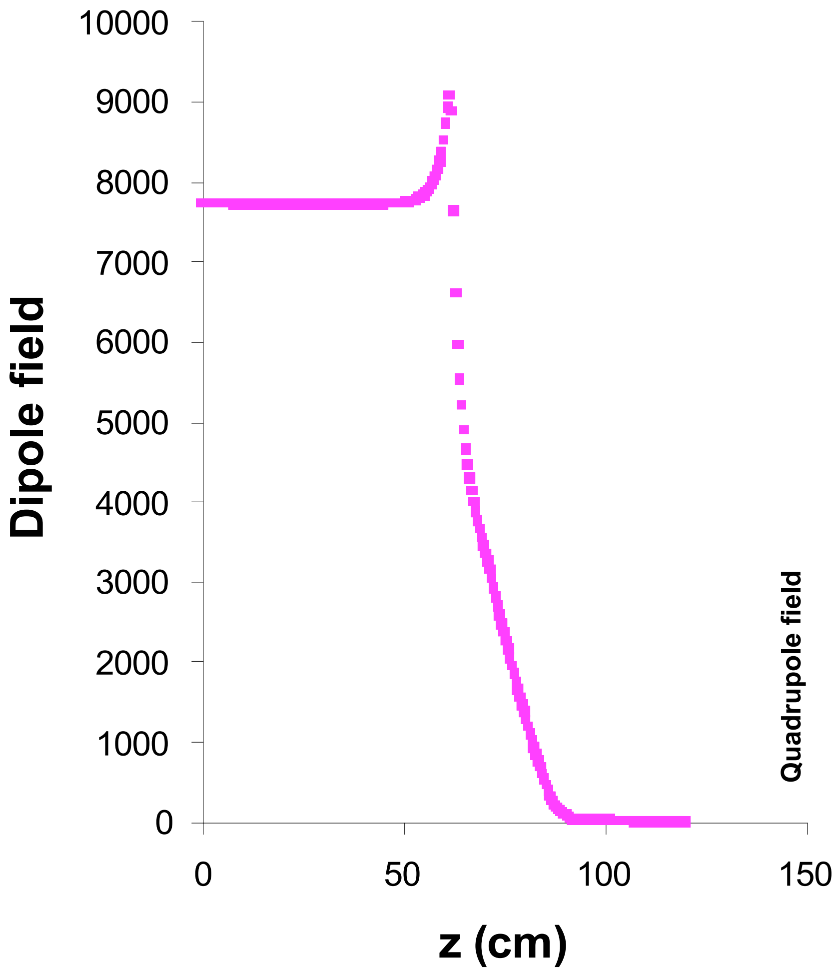


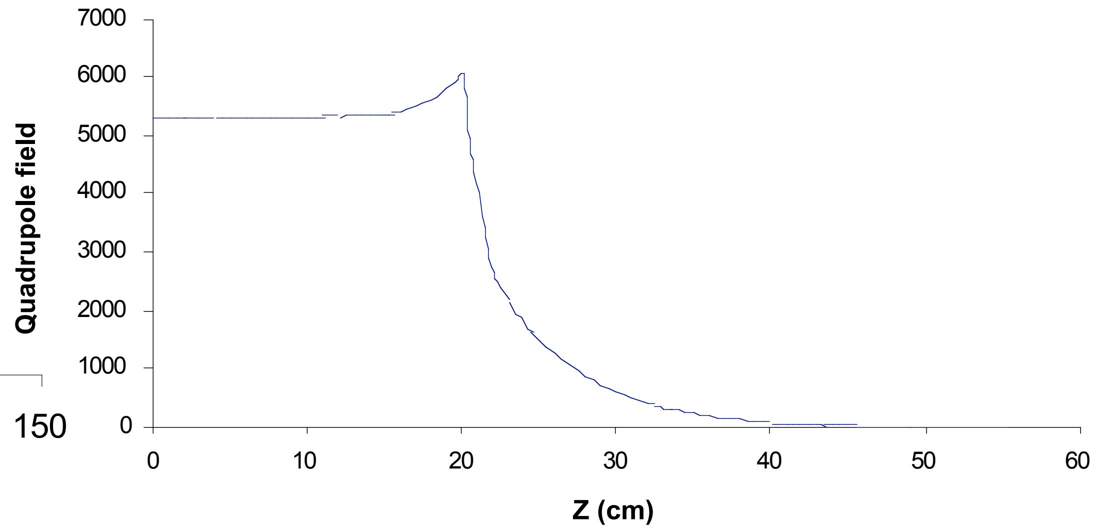


- Drop of the DA without chromatic sextupoles in both cases
- Unacceptable drop below physical aperture for dp/p = -0.02 (right)

Magnet fringe fields

- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big





Dipole fringe field

Using the general z-dependent field expansion, for a straight dipole:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+2-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m}}{(2n)!(2m)!} \binom{m}{l} b_{2n+2m-2l}^{[2l]}$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m-2l}^{[2l+1]}$$

and to leading order:

$$\begin{aligned} B_x &= b_2 xy + O(4) \\ B_y &= b_0 - \frac{1}{2} b_0^{[2]} y^2 + \frac{1}{2} b_2 (x^2 - y^2) + O(4) \\ B_z &= y\, b_0^{[1]} + O(3) \end{aligned}$$

Dipole fringe to leading order gives a sextupole-like effect (vertical chromaticity)

Quadrupole fringe field

General field expansion for a quadrupole magnet:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} \quad .$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l+1]}$$

and to leading order

$$\begin{aligned} B_x &= y\left[b_1 - \frac{1}{12}(3x^2+y^2)b_1^{[2]}\right] + O(5) \\ B_y &= x\left[b_1 - \frac{1}{12}(3y^2+x^2)b_1^{[2]}\right] + O(5) \\ B_z &= xyb_1^{[1]} + O(4) \end{aligned}$$

The quadrupole fringe to leading order has an octupole-like effect

Quadrupole fringe field octupole effect

The hard-edge Hamiltonian (Forest and Milutinovic 1988)

$$H_f = \frac{\pm Q}{12B\rho(1+\frac{\delta p}{p})}(y^3 p_y - x^3 p_x + 3x^2 y p_y - 3y^2 x p_x),$$

First order tune spread for an octupole:

$$\begin{pmatrix} \delta\nu_x \\ \delta\nu_y \end{pmatrix} = \begin{pmatrix} a_{hh} & a_{hv} \\ a_{hv} & a_{vv} \end{pmatrix} \begin{pmatrix} 2J_x \\ 2J_y \end{pmatrix},$$

where the normalized anharmonicities are

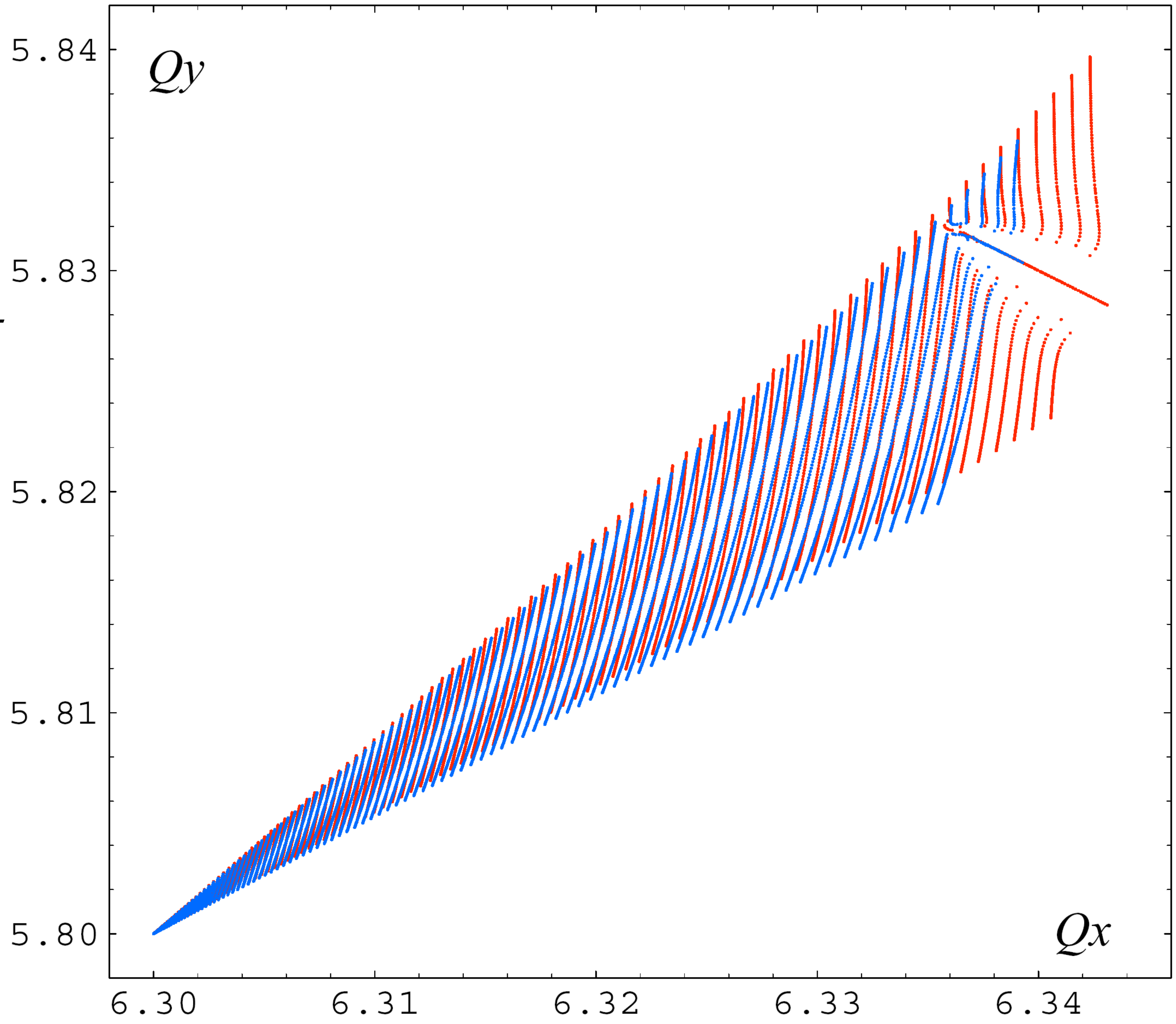
$$a_{hh} = \frac{-1}{16\pi B\rho} \sum_i \pm Q_i \beta_{xi} \alpha_{xi},$$

$$a_{hv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i (\beta_{xi}\alpha_{yi} - \beta_{yi}\alpha_{xi}),$$

$$a_{vv} = \frac{1}{16\pi B\rho} \sum_i \pm Q_i \beta_{yi} \alpha_{yi}.$$

Quadrupole fringe field tune spread

Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field



Frequency map analysis

Quasi-periodic approximation through **NAFF** algorithm

$$f_j'(t) = \sum_{k=1}^{N} a_{j,k} e^{i\omega_{j,k} t}$$

of a complex phase space function $f_j(t) = q_j(t) + ip_j(t)$

defined over $t = \tau$,

for each degree of freedom $j = 1, \ldots, n$ with $\omega_{j,k} = \boldsymbol{k}_j \cdot \boldsymbol{\omega}$

and $a_{j,k} = A_{j,k} e^{i\phi_{j,k}}$

Advantages of NAFF:

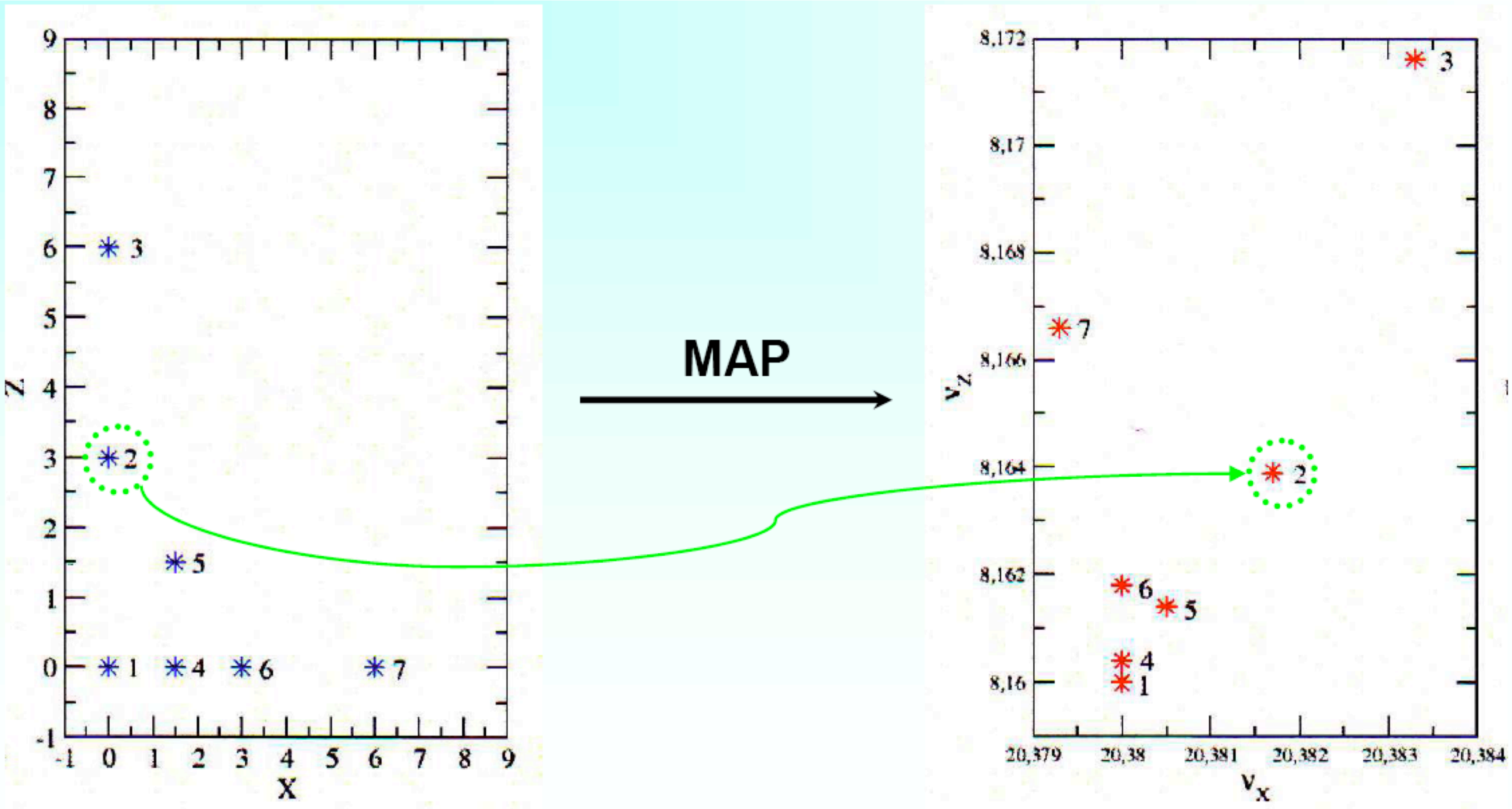
a) Very accurate representation of the "signal" $f_j(t)$ (if quasi-periodic) and thus of the amplitudes $a_{j,k}$

b) Determination of frequency vector $\boldsymbol{\omega} = 2\pi\boldsymbol{\nu} = 2\pi(\nu_1, \nu_2, \ldots, \nu_n)$

with high precision $\frac{1}{\tau^4}$ for Hanning Filter

Building the frequency map

- Choose coordinates (x_i, y_i) with p_x and p_y=0
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram



Aspects of frequency map analysis

- Construction of frequency map

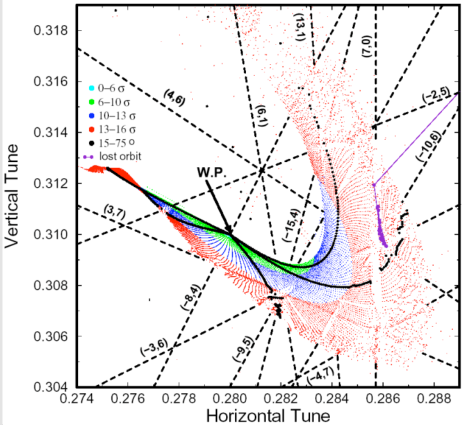
$$\boxed{\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{q}|_{\boldsymbol{p}=\boldsymbol{p}_0} & \longrightarrow & \boldsymbol{\nu} \end{array}}$$

- Determination of tune diffusion vector

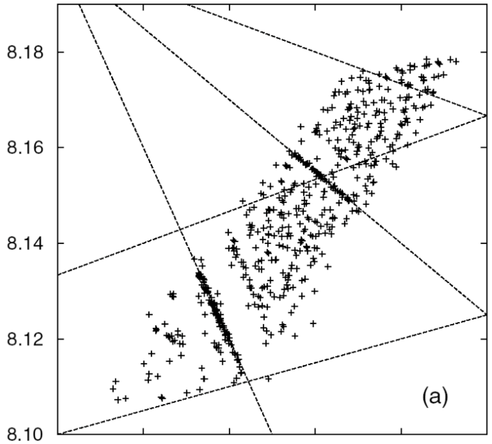
$$\boldsymbol{D}|_{t=\tau} = \boldsymbol{\nu}|_{t\in(0,\tau/2]} - \boldsymbol{\nu}|_{t\in(\tau/2,\tau]}$$

and construction of diffusion map

$$\boxed{\mathcal{D}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{q}|_{\boldsymbol{p}=\boldsymbol{p}_0} & \longrightarrow & \boldsymbol{D} \end{array}}$$



LHC Simulations
Papaphilippou PAC99

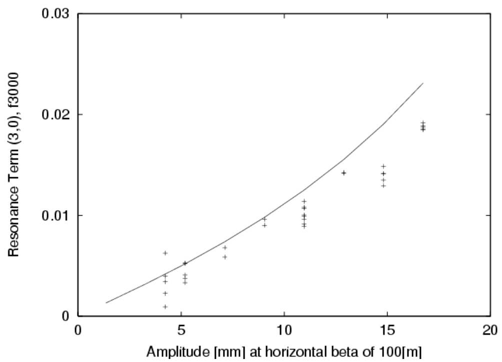


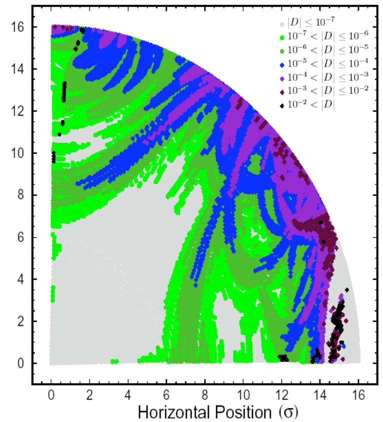
ALS Measurements
Robin et al. PRL2000

- Determination of resonance driving terms associated with amplitudes $a_{j,k}$

Bengtsson PhD thesis CERN88-05

SPS Measurements
Bartolini et al. PAC99

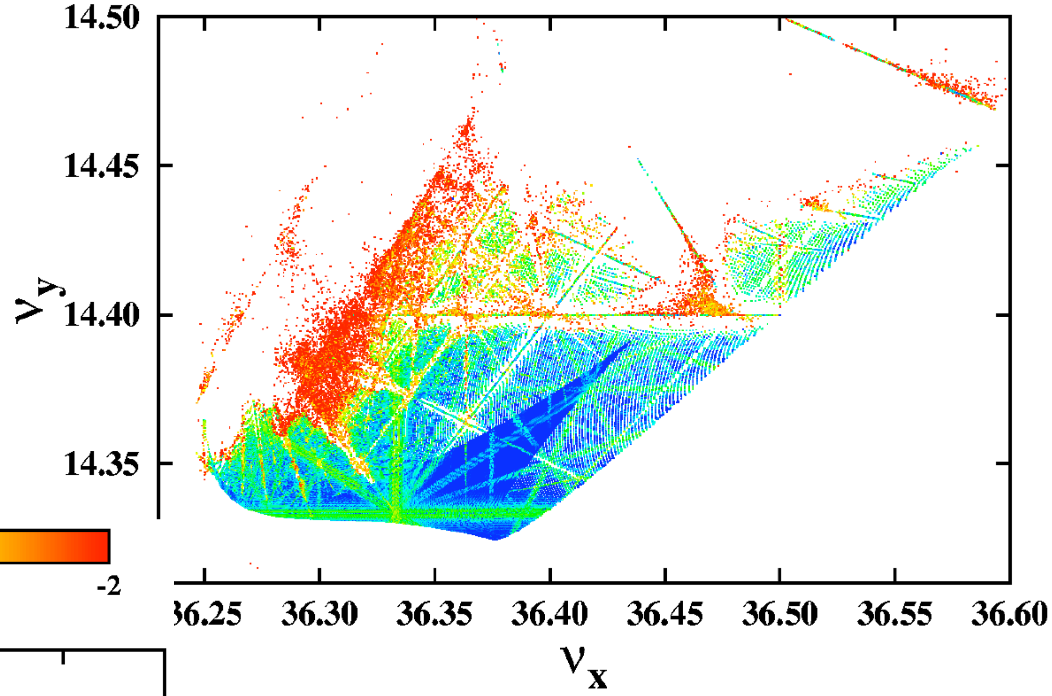


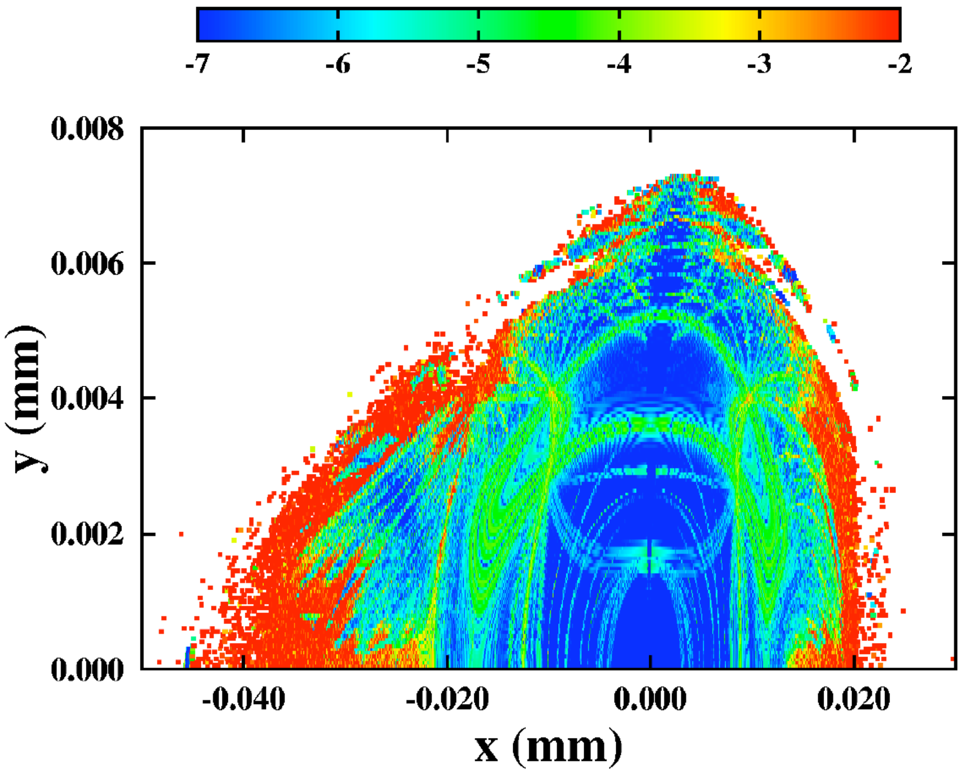


LHC Simulations
Papaphilippou PAC99

Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)





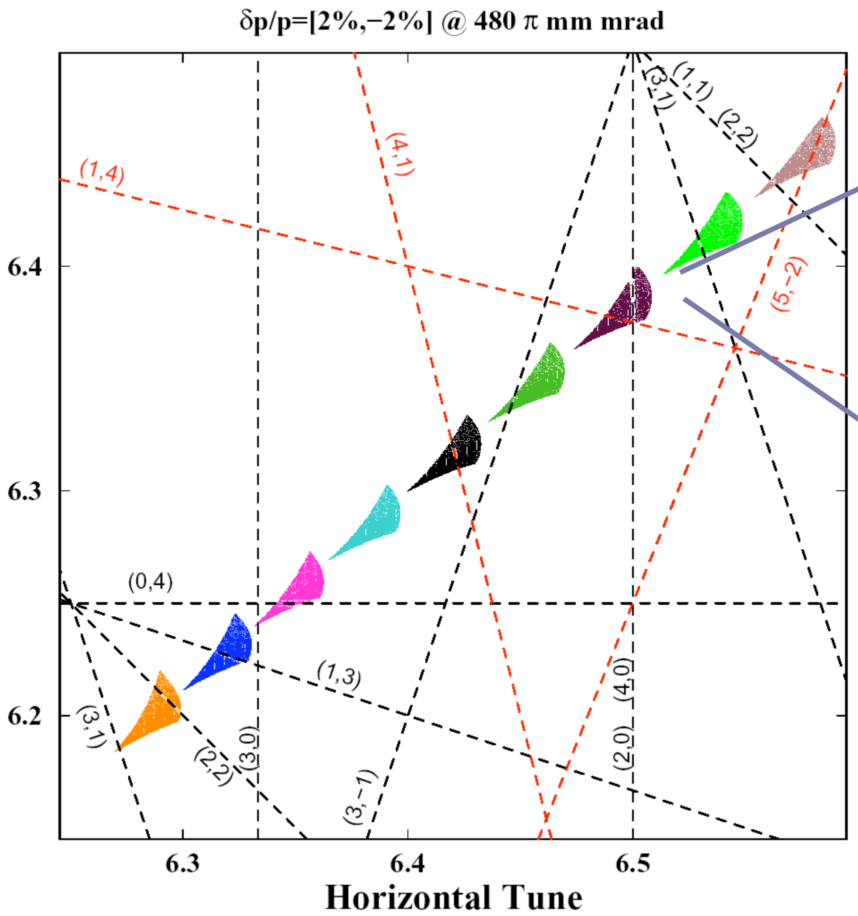
- Resonances appear as distorted lines in frequency space (or curves in initial condition space
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine

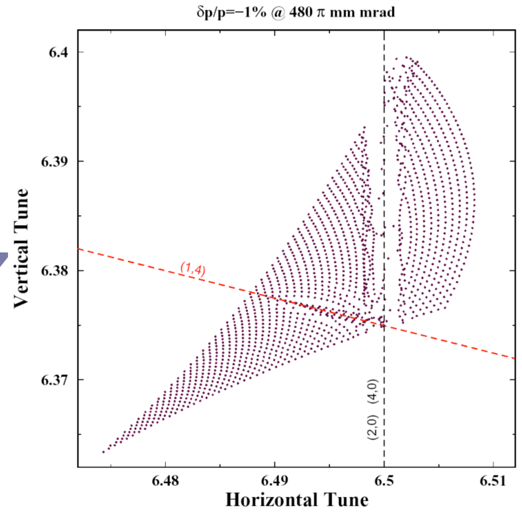
Example for the SNS ring: Working point (6.4,6.3)

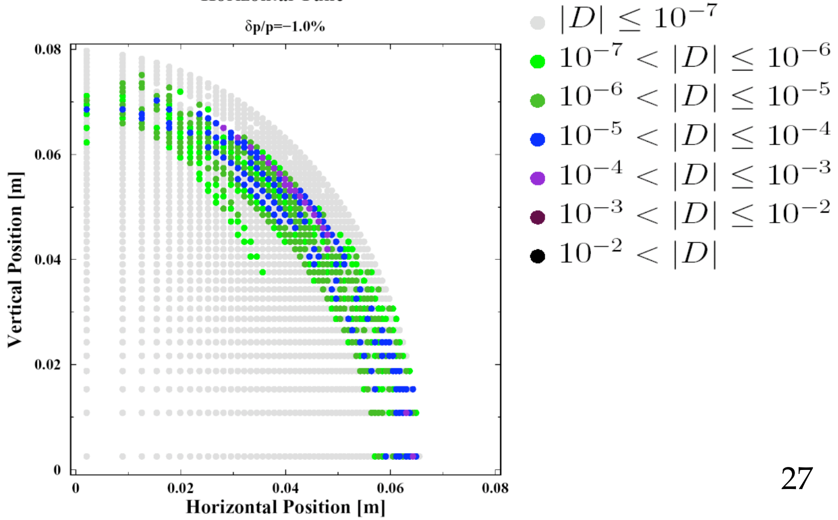
- Integrate a large number of particles
- Calculate the tune with refined Fourier analysis
- Plot points to tune space

$$\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (I_x, I_y)|_{p_x,p_y=0}, & \longrightarrow & (\nu_x, \nu_y) \end{array}$$

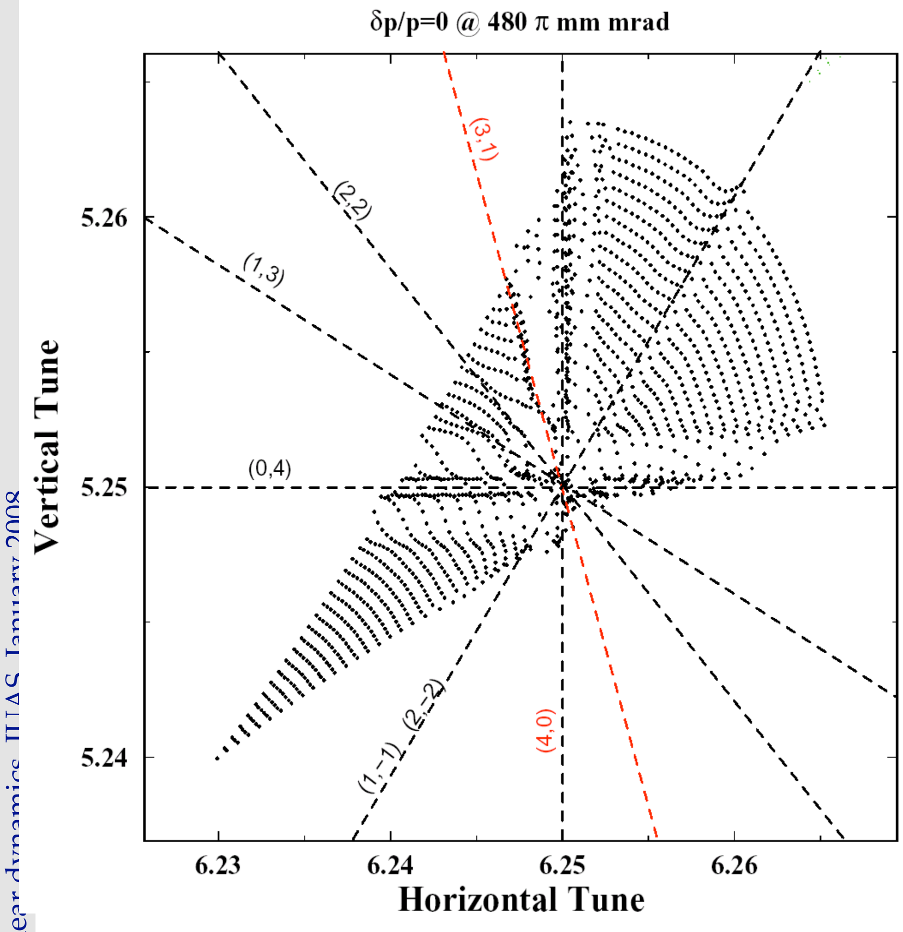
SNS Working Point (Q_x,Q_y)=(6.4,6.3)

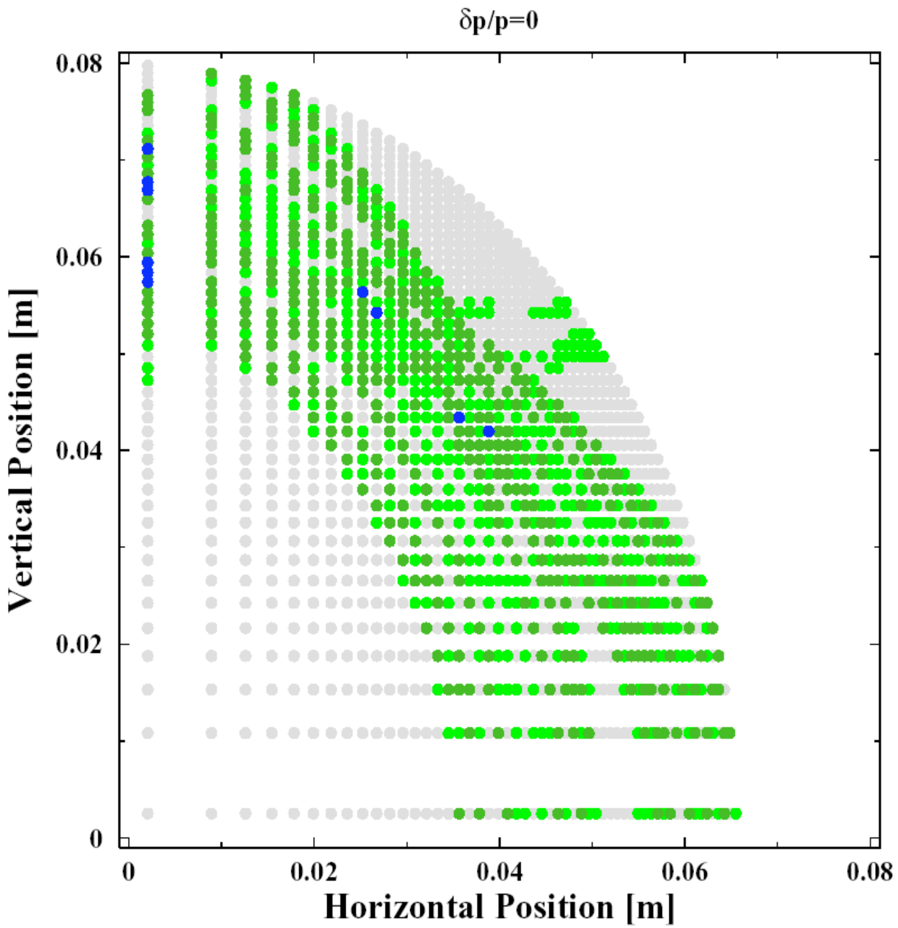






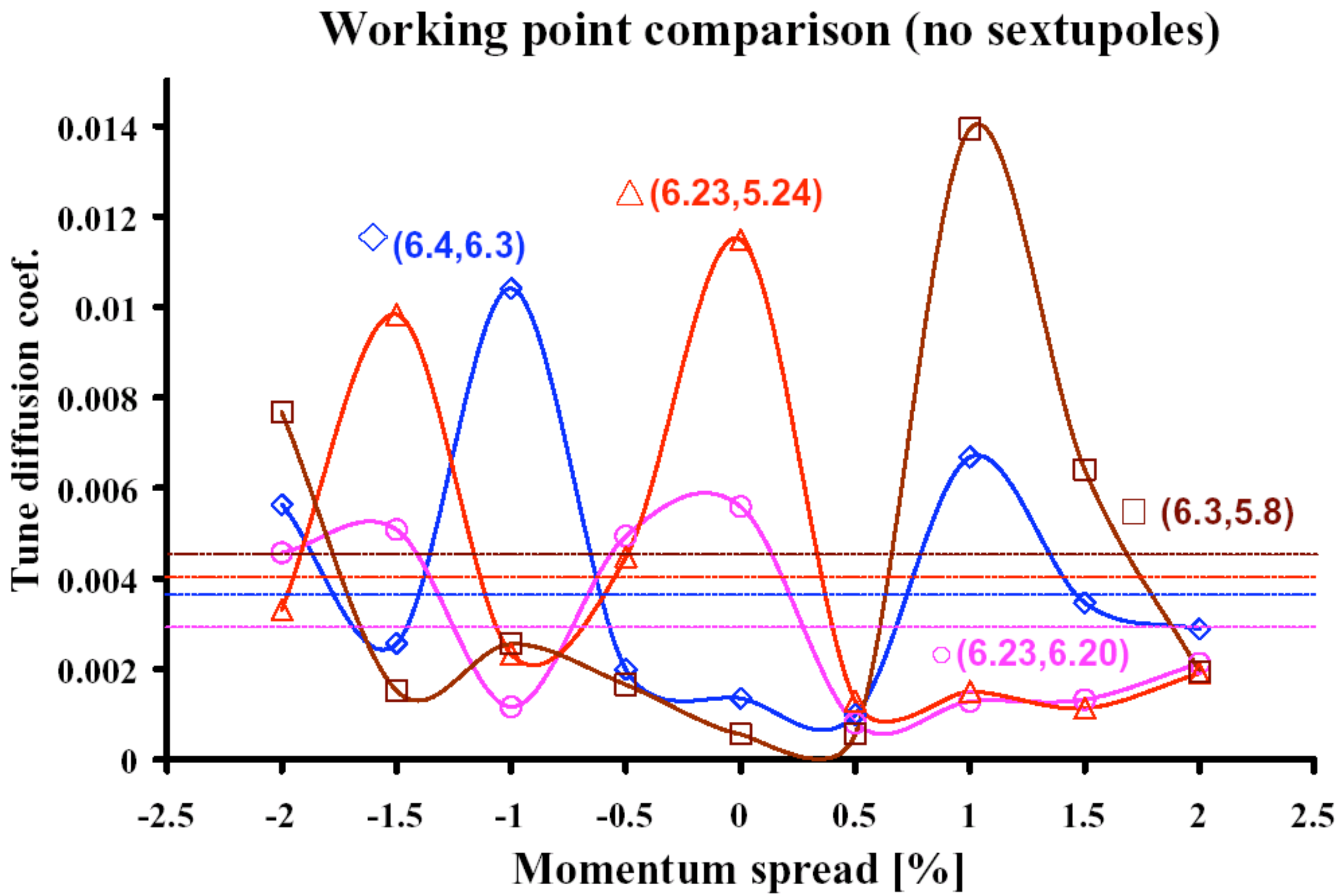
SNS Working point (6.23,5.24)





Working Point Comparison

Tune Diffusion quality factor $$D_{QF} = \left\langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$



References

- O. Bruning, Non-linear dynamics, JUAS courses, 2006.