Lattice Design

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Outline

- Periodic lattices in circular accelerators
  - Periodic solutions for beta function and dispersion
  - Symmetric solution
- FODO cell
  - Betatron functions and phase advances
  - Optimum betatron functions
  - General FODO cell and stability
  - Solution for dispersion
  - Dispersion supressors
- General periodic solutions for the dispersion
- Tune and Working point
- Matching the optics

Periodic solutions

- Consider two points $\mathbf{s_0}$ and $\mathbf{s_1}$ for which the magnetic structure is repeated.
- The optical function follow periodicity conditions

$$\beta_0 = \beta(s_0) = \beta(s_1)\ ,\quad \alpha_0 = \alpha(s_0) = \alpha(s_1)$$
$$D_0 = D(s_0) = D(s_1)\ ,\ D'_0 = D'(s_0) = D'(s_1)$$

- The beta matrix at this point is $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$
- Considering the transfer matrix between $\mathbf{s_0}$ and $\mathbf{s_1}$ $\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$$\mathcal{B}_0 = \mathcal{M}_{0\to 1}\cdot\mathcal{B}_0\cdot\mathcal{M}_{0\to 1}^T \Rightarrow \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}\begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$
$$\alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

with the condition $2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2 > 0$

Periodic solutions for the dispersion

- Consider the 3x3 matrix for propagating dispersion between $\mathbf{s_0}$ and $\mathbf{s_1}$

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

- Solve for the dispersion and its derivative to get

$$D_0' = \frac{m_{21}m_{13} + m_{23}(1 - m_{11})}{2 - m_{11} - m_{22}}$$

$$D_0 = \frac{m_{12}D_0' + m_{13}}{1 - m_{11}}$$

with the conditions

$$m_{11} + m_{22} \neq 2 \text{ and } m_{11} \neq 1$$

Symmetric solutions

- Consider two points $\mathbf{s_0}$ and $\mathbf{s_1}$ for which the lattice is mirror symmetric
- The optical function follow periodicity conditions

$$\alpha(s_0) = \alpha(s_1) = 0$$
$$D'(s_0) = D'(s_1) = 0$$

- The beta matrices at these points is $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$ $\mathcal{B}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$
- Considering the transfer matrix between $\mathbf{s_0}$ and $\mathbf{s_1}$

$$\mathcal{B}_1 = \mathcal{M}_{0\to1} \cdot \mathcal{B}_0 \cdot \mathcal{M}_{0\to1}^T \Rightarrow \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \sqrt{-\frac{m_{12}m_{22}}{m_{21}m_{11}}} \text{ and } \beta_1 = -\frac{1}{\beta_0}\frac{m_{12}}{m_{21}}$$

with the condition $$\frac{m_{12}}{m_{21}} < 0 \text{ and } \frac{m_{22}}{m_{11}} > 0$$

Symmetric solutions for the dispersion

- Consider the 3x3 matrix for propagating dispersion between $\mathbf{s_0}$ and $\mathbf{s_1}$

$$\begin{pmatrix} D(s_1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_0) \\ 0 \\ 1 \end{pmatrix}$$

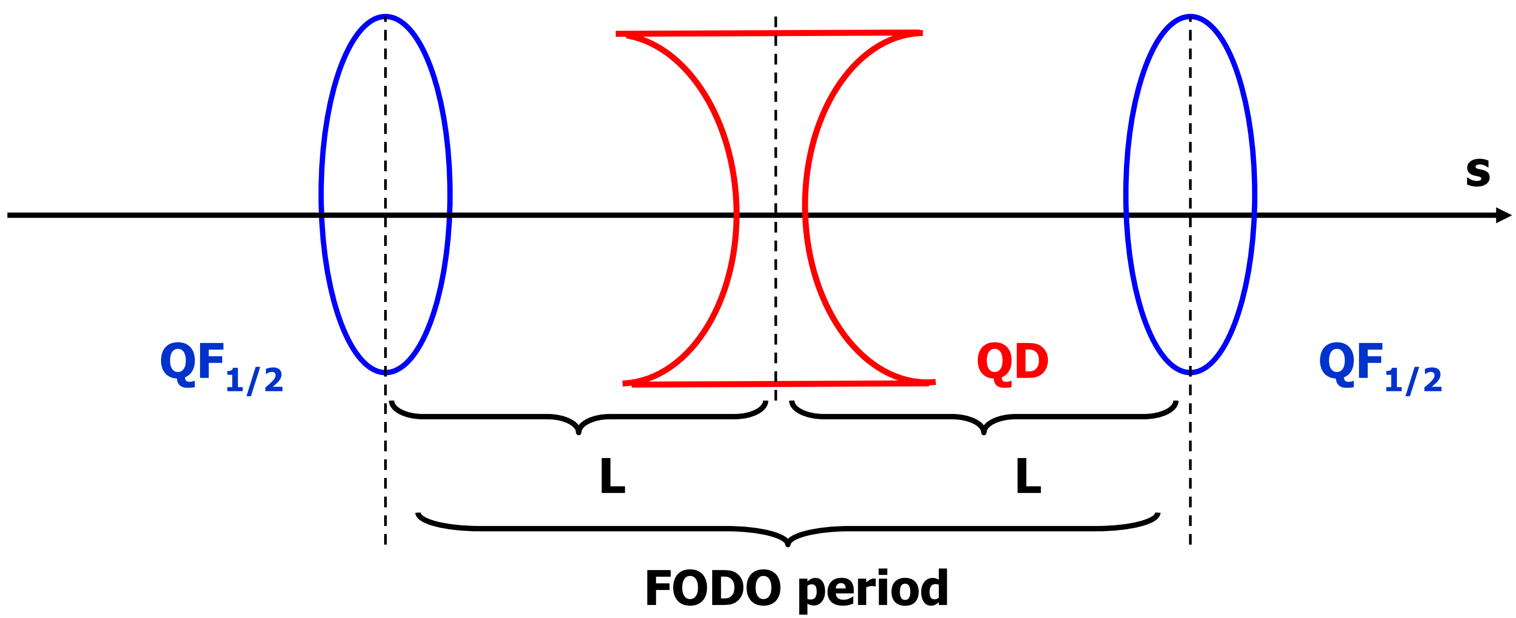
- Solve for the dispersion in the two locations

$$\begin{aligned} D(s_0) &= -\frac{m_{23}}{m_{21}} \\ D(s_1) &= -\frac{m_{11} m_{23}}{m_{21}} + m_{13} \end{aligned}$$

- Imposing certain values for beta and dispersion, quadrupoles can be adjusted in order to get a solution

FODO Cell

- FODO is the simplest basic structure
  - Half focusing quadrupole (F) + Drift (O) + Defocusing quadrupole (D) + Drift (O)
  - Dipoles can be added in drifts for bending
  - Periodic lattice with mirror symmetry in the center
  - Cell period from center to center of focusing quadrupole
  - The most common structure is accelerators



FODO transfer matrix

- Restrict study in thin lens approximation for simplicity
- FODO symmetric from any point to any point separated by 2L
- Useful to start and end at center of QF or QD, due to mirror symmetry
- The transfer matrix is

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

and we have

$$\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ \frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$

where we set $f_{\mathrm{QF}} = -f_{\mathrm{QD}} = f$ for a **symmetric FODO**

- Note that diagonal elements are equal due to mirror symmetry

$$\mathcal{M}_{\mathrm{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix} \quad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

Betatron function for a FODO

- By using the formulas for the symmetric optics functions

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}} \quad \text{and} \quad \alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

we get the beta on the center of the focusing quad

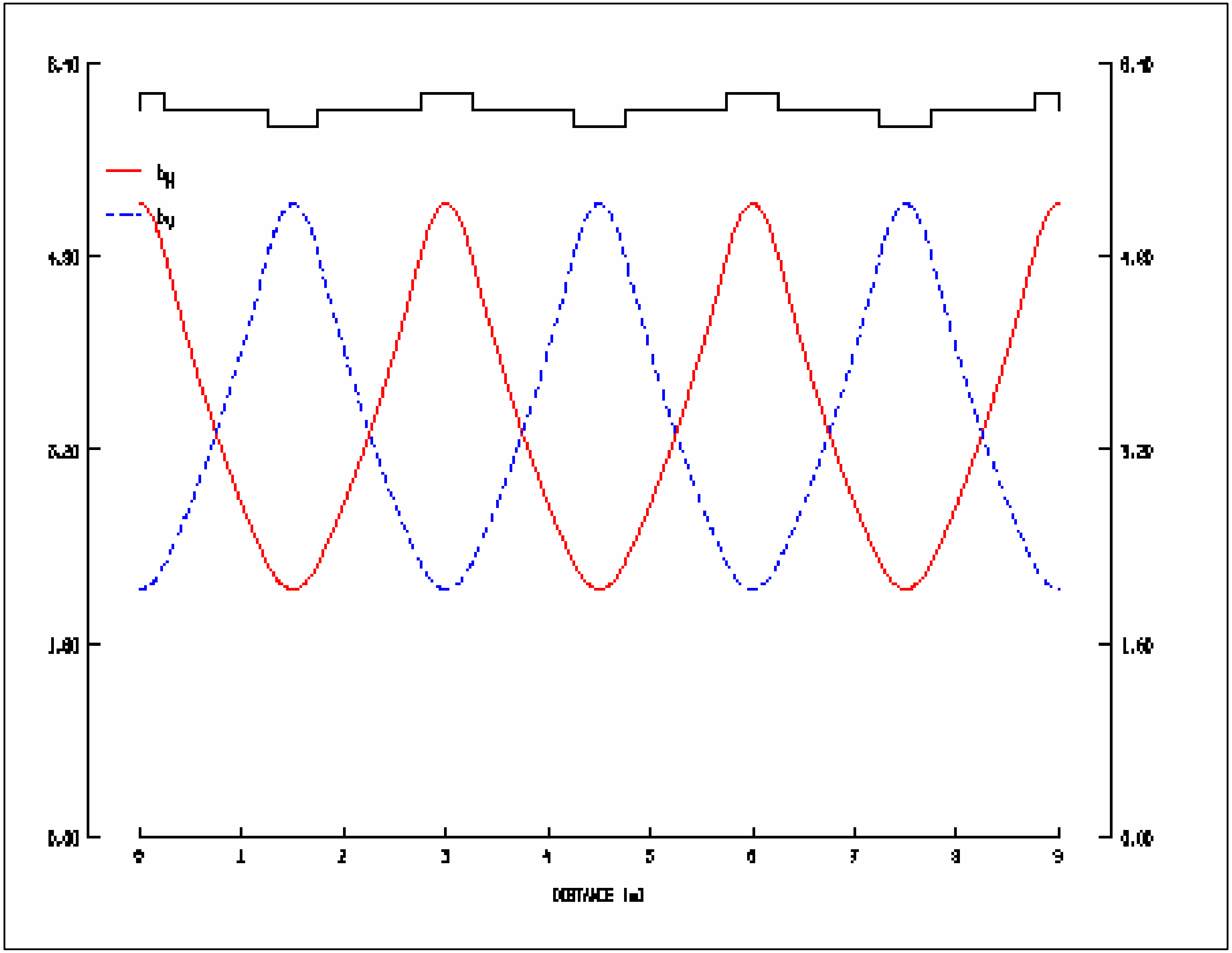
$$\beta^+ = L\frac{\kappa(\kappa + 1)}{\sqrt{\kappa^2 - 1}} \quad \text{with} \quad \kappa = f/L > 1$$

- Starting in the center of the defocusing quad (simply setting **f** to **-f**)

$$\beta^- = L\frac{\kappa(\kappa - 1)}{\sqrt{\kappa^2 - 1}}$$

- Solutions for both horizontal and vertical plane
  - In the center of QF $\beta_x = \beta^+$ and $\beta_y = \beta^-$
  - In the center of QD $\beta_x = \beta^-$ and $\beta_y = \beta^+$
- Knowing the beta functions at one point, their evolution can be determined through the FODO cell

- Betatron functions evolution in a FODO cell



Phase advance for a FODO

- For a symmetric cell, the transfer matrix can be written as

$$\mathcal{M}_{\text{sym}} = \begin{pmatrix} \cos\phi & \beta\sin\phi \\ -\frac{1}{\beta}\sin\phi & \cos\phi \end{pmatrix}$$

- So the phase advance is

$$\cos\phi = 1 - 2\frac{L^2}{f^2} = \frac{\kappa^2 - 2}{\kappa^2} \quad \text{or} \quad \sin\frac{\phi}{2} = \frac{1}{\kappa}$$

- This imposes the condition $\kappa > 1$ which means that the focal length should be smaller than the distance between quads
- For $\kappa \to 1$, the beta function becomes infinite, so in between there should be a minimum

Optimum betatron functions in a FODO

- Start from the solution for beta in the focusing quad

$$\beta^+ = L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}}$$

- Take the derivative to vanish $\frac{d\beta^+}{d\kappa} = 0 \rightarrow \kappa_0^2 - \kappa_0 - 1 = 0$
- The solution for the focusing strength is

$$\kappa_0 = \frac{1}{2} \pm \sqrt{\frac{1}{4}+1} \approx 1.6180$$

- So the optimum phase advance is $\phi_0 \approx 76.345^\circ$
- This solution however cannot minimize the betatron function in both planes
- It is good only for flat beams $\epsilon_x >> \epsilon_y$ or $\epsilon_y >> \epsilon_x$

Optimum betatron functions for round beams

- Consider a round beam $\epsilon_x \approx \epsilon_y$
- The maximum beam acceptance is obtained by minimizing quadratic sum of the envelopes

$$E_x^2 + E_y^2 = \epsilon_x \beta_x + \epsilon_y \beta_y \approx \epsilon(\beta_x + \beta_y)$$

- The minimum is determined by $\frac{d}{d\kappa}(\beta^+ + \beta^-) = 0$
- The minimum is reached for $\kappa_0 = \sqrt{2}$

  and the optimum phase is $\phi_0 = 90°$
- The betatron functions are $\beta_{\rm opt}^{\pm} = L(2 \pm \sqrt{2})$
- In order to fit an aperture of radius **R**

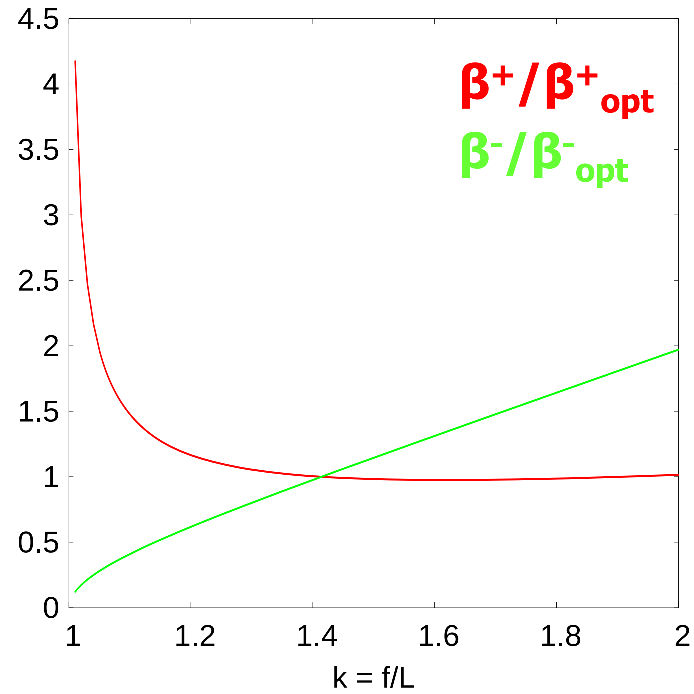
$$E_x^2 + E_y^2 = R^2 = \epsilon(\beta^+ + \beta^-) = \epsilon \ 4L$$

- The maximum emittance is $\epsilon_{\rm max} = \frac{R^2}{4L}$

- Scaling of the betatron functions with respect to the optimum values

$$\frac{\beta^+}{\beta^+_{\mathrm{opt}}} = \frac{\kappa(\kappa+1)}{(2+\sqrt{2})\sqrt{\kappa^2-1}} \quad \text{and} \quad \frac{\beta^-}{\beta^-_{\mathrm{opt}}} = \frac{\kappa(\kappa-1)}{(2-\sqrt{2})\sqrt{\kappa^2-1}}$$

- Scaling is independent of **L**
- It only depends on the ratio of the focal length and **L**
- The distance can be adjusted as a free parameter
- As the maximum beta functions are scaled linearly with **L**
- The maximum beam size in a FODO cell scales like with $\sqrt{L}$



Periodic lattices' stability criterion revisited

- Consider a general periodic structure of length 2L which contains **N** cells. The transfer matrix can be written as

$$\mathcal{M}(s+N\cdot 2L|s)=\mathcal{M}(s+2L|s)^N$$

- The periodic structure can be expressed as

$$\mathcal{M}=\mathcal{I}\cos\mu+\mathcal{J}\sin\mu$$

with $\mathcal{J}=\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}$.

- Note that because $det(\mathcal{M})=1\rightarrow\beta\gamma-a^2=1$
- Note also that $\mathcal{J}^2=-\mathcal{I}$
- By using **de Moivre's formula**

$$\mathcal{M}^N=(\mathcal{I}\cos\mu+\mathcal{J}\sin\mu)^N=\mathcal{I}\cos(N\mu)+\mathcal{J}\sin(N\mu)$$

- We have the following general stability criterion

$$|\mathrm{Trace}(\mathcal{M}^N)|=2\cos(N\phi)<2$$

General FODO cell

- So far considered transformation matrix for equal strength quadrupoles
- The general transformation matrix for a FODO cell

$$\mathcal{M}_{1/2} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1-\frac{L}{f_2} \end{pmatrix}$$

with $$\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\, f_2}$$

- Multiplication with the reverse matrix $\mathcal{M}^r_{1/2} = \begin{pmatrix} 1-\frac{L}{f_2} & L \\ -\frac{1}{f^\star} & 1-\frac{L}{f_1} \end{pmatrix}$ gives

$$\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1-\frac{2L}{f^\star} & 2L(1-\frac{L}{f_2}) \\ -\frac{2}{f^\star}(1-\frac{L}{f_1}) & 1-\frac{2L}{f^\star} \end{pmatrix}$$

Stability for a general FODO cell

- Setting $\frac{L}{f_1} = F$ and $-\frac{L}{f_2} = D$ we have hat the transfer matrix for a half cell is

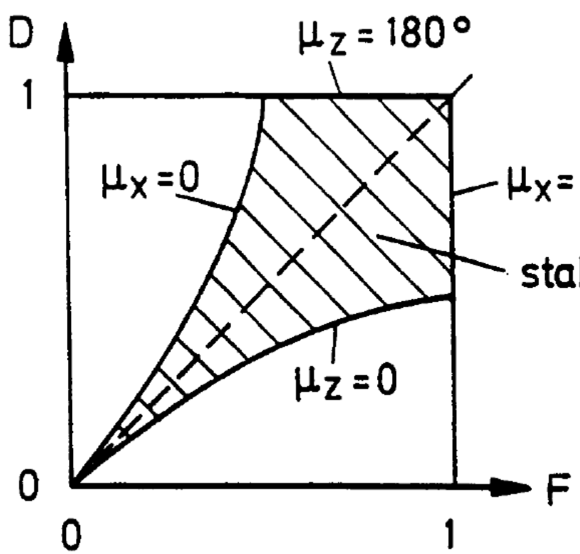
$$\mathcal{M}_{1/2\mathrm{FODO}} = \begin{pmatrix} 1-F & L \\ -\frac{1}{L}(F-D+FD) & 1+D \end{pmatrix}$$

- Equating this with the betatron transfer matrix we have

$$\mathcal{M}_{1/2FODO} = \begin{pmatrix} \sqrt{\frac{\beta-}{\beta+}}\cos\mu/2 & \sqrt{\beta+\beta-}\sin\mu/2 \\ -\frac{\sin\mu/2}{\sqrt{\beta-\beta+}} & \sqrt{\frac{\beta+}{\beta-}}\cos\mu/2 \end{pmatrix}$$

$$0 < F - D + FD = \sin^2\mu^+/2 < 1$$

$$0 < D - F + FD = \sin^2\mu^-/2 < 1$$



- The limits of the stable region give a **necktie**

$$\sin^2\mu^+/2 = 0 \to F = \frac{D}{1+D} \qquad \sin^2\mu^+/2 = 1 \to F = 1$$

$$\sin^2\mu^-/2 = 0 \to D = \frac{F}{1+F} \qquad \sin^2\mu^-/2 = 1 \to D = 1$$

3X3 FODO cell matrix

- Insert a sector dipole in between the quads and consider **θ=L/ρ<<1**
- Now the transfer matrix is $\mathcal{M}_{\rm HFODO} = \mathcal{M}_{\rm HQF} \cdot \mathcal{M}_{\rm sector} \cdot \mathcal{M}_{\rm HQD}$

  which gives

$$\mathcal{M}_{\rm HFODO} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L^2}{2\rho} \\ 0 & 1 & \frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  and after multiplication

$$\mathcal{M}_{\rm HFODO} = \begin{pmatrix} 1 - \frac{L}{f} & L & \frac{L^2}{(2\rho)} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} & \frac{L}{\rho}(1 + \frac{L}{2f}) \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion in a FODO cell

- Consider mirror symmetry conditions, i.e. the dispersion derivative vanishes in the middle of quads

$$\begin{pmatrix} \eta^- \\ 0 \\ 1 \end{pmatrix} = \mathcal{M}_{\text{HFODO}} \begin{pmatrix} \eta^+ \\ 0 \\ 1 \end{pmatrix}$$

- Solving for the dispersion in the entrance and exit

$$\eta^+ = \frac{L^2}{2\rho}\kappa(2\kappa+1) \;\text{ and }\; \eta^- = \frac{L^2}{2\rho}\kappa(2\kappa-1) \;\text{ with }\; \kappa = f/L$$

- We choose an optimum reference lattice where $\kappa_0 = \sqrt{2}$

$$\eta^+_{\text{opt}} = \frac{L^2}{2\rho}(4+\sqrt{2}) \;\text{ and }\; \eta^-_{\text{opt}} = \frac{L^2}{2\rho}(4-\sqrt{2})$$

and the ratio $\dfrac{\eta^+}{\eta^+_{\text{opt}}} = \dfrac{\kappa(2\kappa+1)}{(4+\sqrt{2})}$ and $\dfrac{\eta^-}{\eta^-_{\text{opt}}} = \dfrac{\kappa(2\kappa-1)}{(4-\sqrt{2})}$

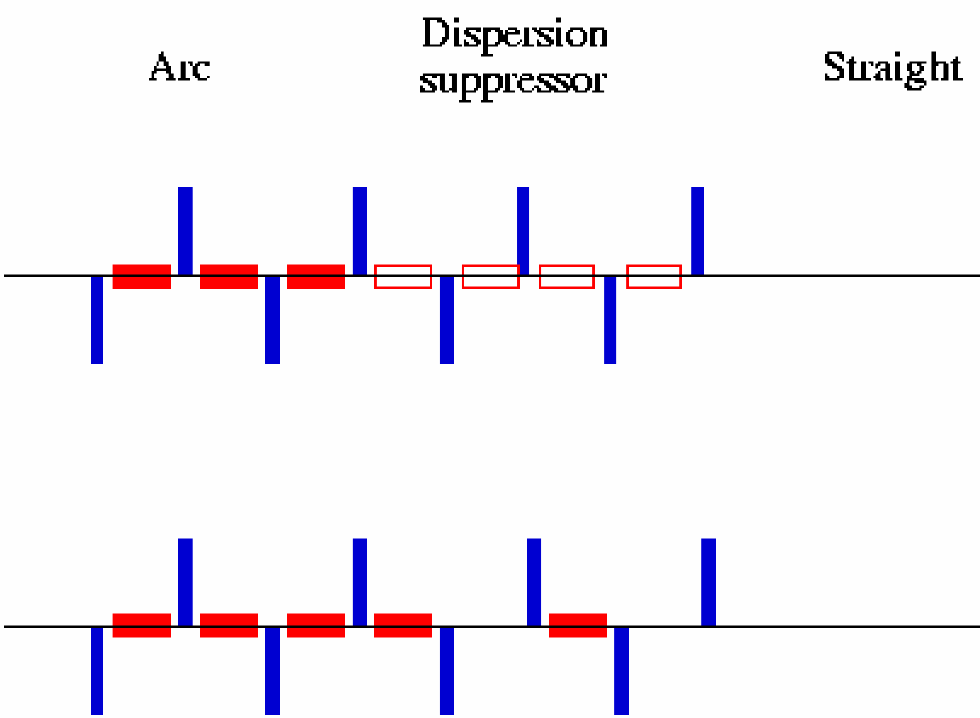
Dispersion suppressors

- Dispersion has to be eliminated in special areas like injection, extraction or interaction points (orbit independent to momentum spread)
- Use **dispersion suppressors**
- Two methods for suppressing dispersion
  - Eliminate two dipoles in a FODO cell (missing dipole)
  - Set last dipoles with different bending angles

$$\theta_1 = \theta\left(1 - \frac{1}{4\sin^2 \mu_{\mathrm{HFODO}}}\right)$$

$$\theta_2 = \frac{\theta}{4\sin^2 \mu_{\mathrm{HFODO}}}$$

  - For equal bending angle dipoles the FODO phase advance should be equal to $\boldsymbol{\pi/2}$



General solution for the dispersion

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ , \quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ , \quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int \frac{ds}{\beta(s)}$$

- The Hill's equations are written $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix} \cos(\nu\phi) \\ -\sin(\nu\phi) \end{pmatrix}$$

- For the dispersion solution $\mathcal{U} = \frac{D}{\sqrt{\beta}}\frac{\Delta P}{P}$, we have to consider the inhomogeneous equation in Floquet variables $\frac{d^2 D}{d\phi^2} + \nu^2 D = -\frac{\nu^2\beta^{3/2}}{\rho(s)}$
- This is a forced harmonic oscillator with solution

$$D(s) = \frac{\sqrt{\beta(s)}\nu}{2\sin(\pi\nu)}\oint \frac{\sqrt{\beta(\sigma)}}{\rho(\sigma)}\cos[\nu(\phi(s) - \phi(\sigma) + \pi)]d\sigma$$

- Note the **resonance conditions** for integer tunes!!!

Tune and working point

- In a ring, the **tune** is defined from the 1-turn phase advance

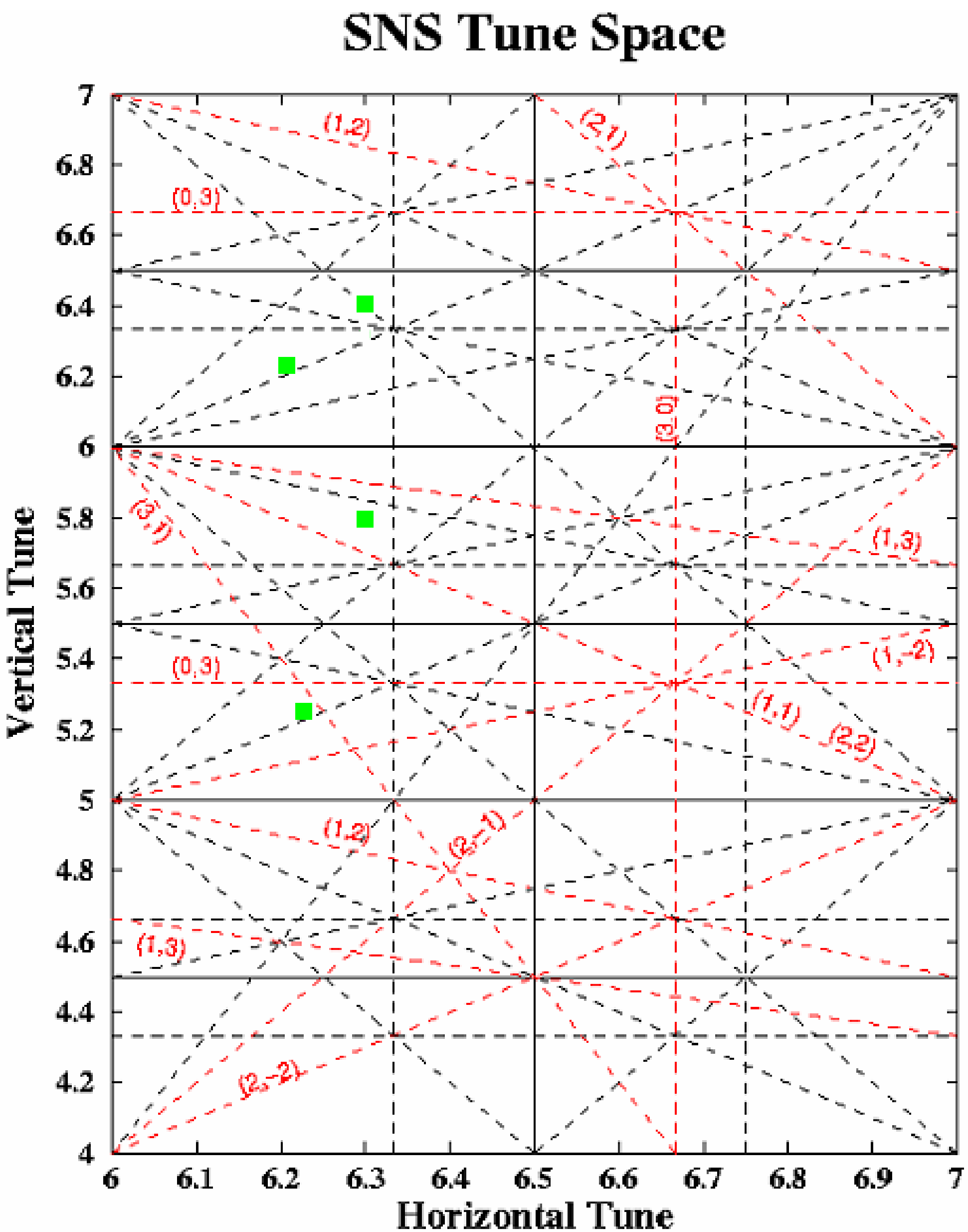
$$Q_{x,y} = \frac{1}{2\pi}\oint \frac{ds}{\beta_{x,y}(s)}$$

i.e. number betatron oscillations per turn
- Taking the average of the betatron tune around the ring we have in **smooth approximation**

$$2\pi Q = \frac{C}{\langle\beta\rangle} \rightarrow Q = \frac{R}{\langle\beta\rangle}$$

- Extremely useful formula for deriving scaling laws
- The position of the tunes in a diagram of horizontal versus vertical tune is called a **working point**
- The tunes are imposed by the choice of the quadrupole strengths
- One should try to avoid **resonance conditions**

Example: SNS Ring Tune Space



Tunability: 1 unit in horizontal, 3 units in vertical (2 units due to bump/chicane perturbation)

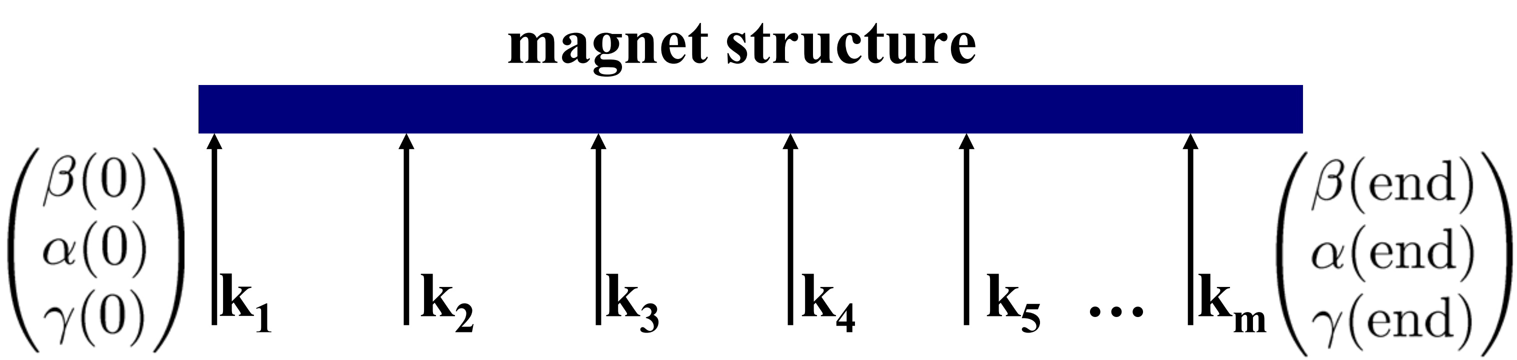
– **Structural resonances (up to 4th order)**

– **All other resonances (up to 3rd order)**

- **Working points considered**
  - **(6.30,5.80) - Old**
  - **(6.23,5.24)**
  - **(6.23,6.20) - Nominal**
  - **(6.40,6.30) - Alternative**

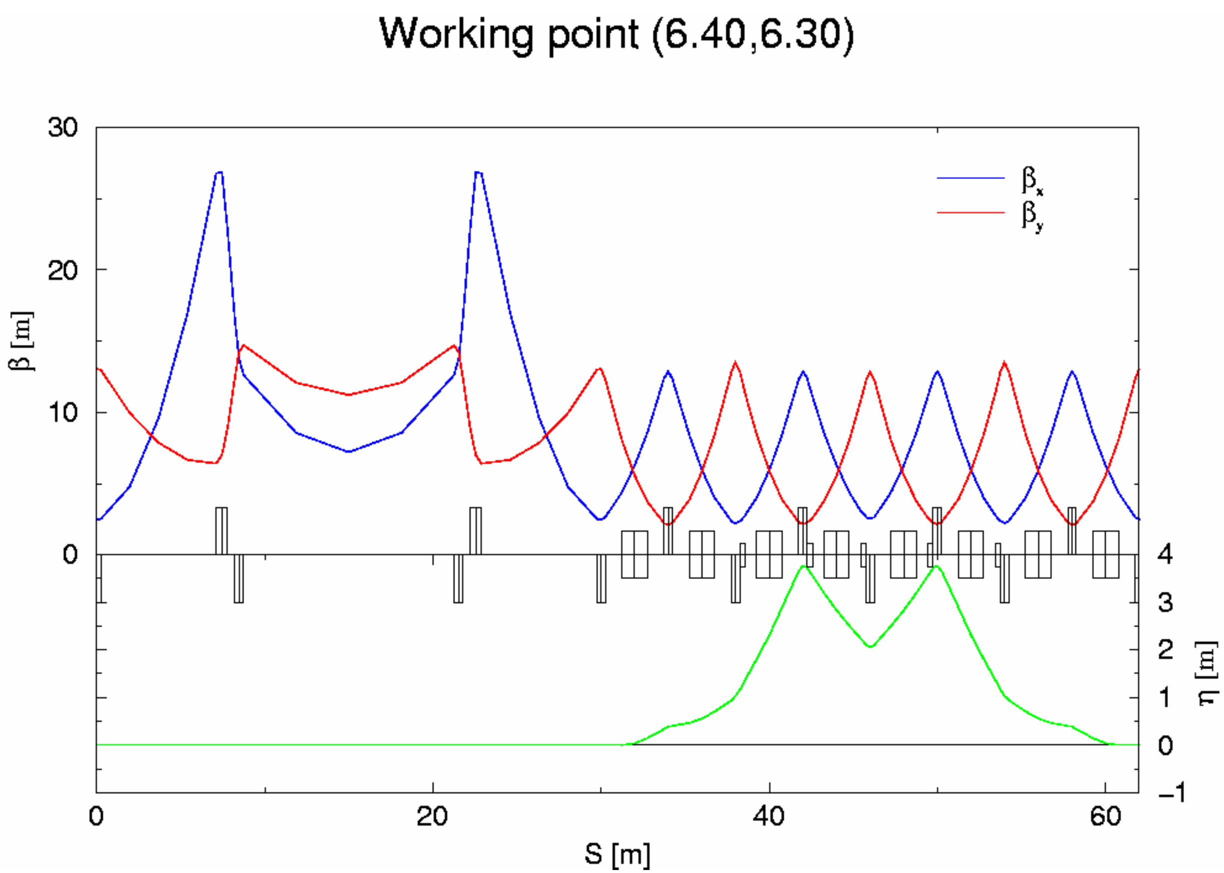
Matching the optics

- Optical function at the **entrance** and **end** of accelerator may be fixed (pre-injector, or experiment upstream)
- Evolution of optical functions determined by magnets through transport matrices
- Requirements for aperture constrain optics functions all along the accelerator
- The procedure for choosing the quadrupole strengths in order to achieve all optics function constraints is called **matching of beam optics**
- Solution is given by numerical simulations with dedicated programs (MAD, TRANSPORT, SAD, BETA, BEAMOPTICS) through multi-variable minimization algorithms

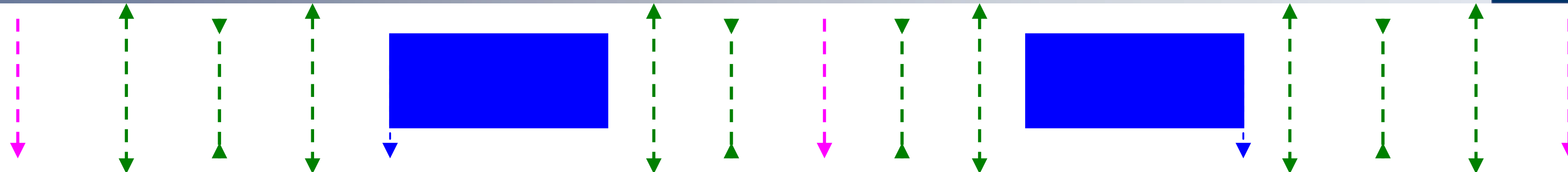


Matching example – the SNS ring

- First find the strengths of the two arc quadrupole families to get an horizontal phase advance of **2π** and using the vertical phase advance as a parameter
- Then match the straight section with arc by using the two doublet quadrupole families and the matching quad at the end of the arc in order to get the correct tune without exceeding the maximum beta function constraints
- Retune arc quads to get correct tunes
- Always keep beta, dispersion within acceptance range and quadrupole strength below design values



ESRF storage ring lattice upgrade



β_{ID1} , $\alpha_{ID1} = 0$, $\eta_{ID1} = \sqrt{\beta_{ID1}\,\mathcal{H}}$, $\eta'_{ID1} = 0$

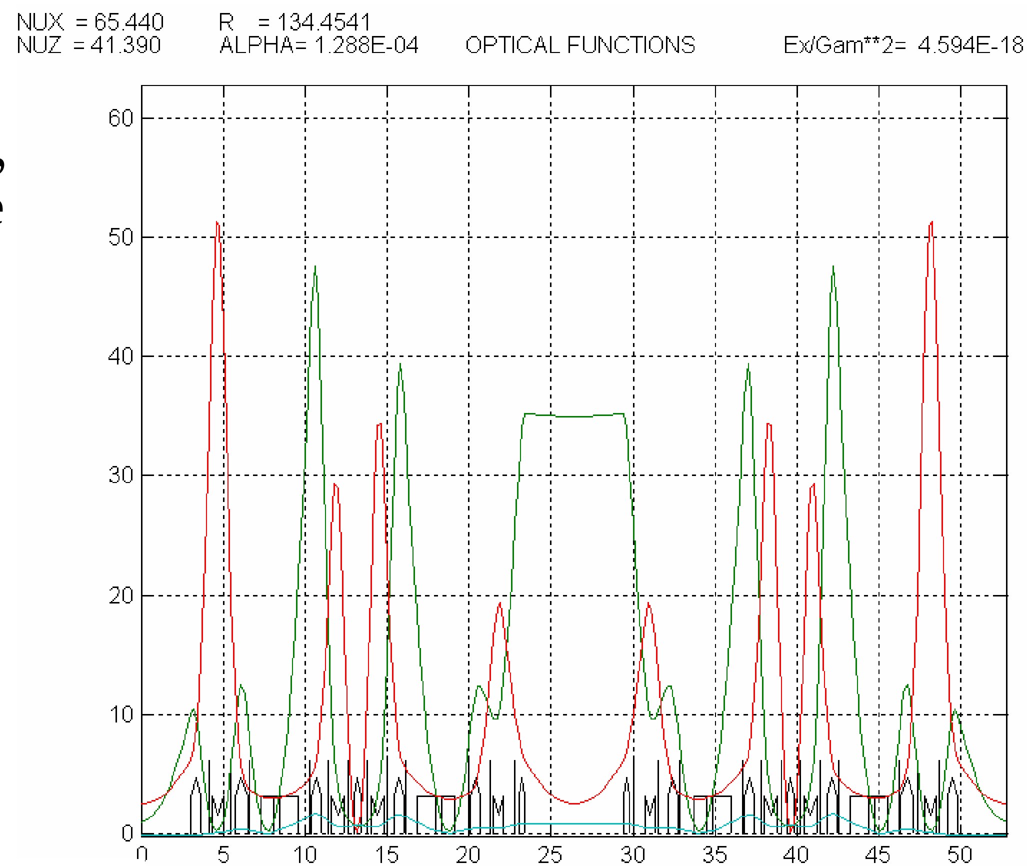
β_0 , α_0 , η_0 , η'_0

β_{SP} , $\alpha_{SP} = 0$, $\eta_{SP} = \sqrt{\beta_{SP}\,\mathcal{H}}$, $\eta'_{SP} = 0$

β_0 , α_0 , η_0 , η'_0

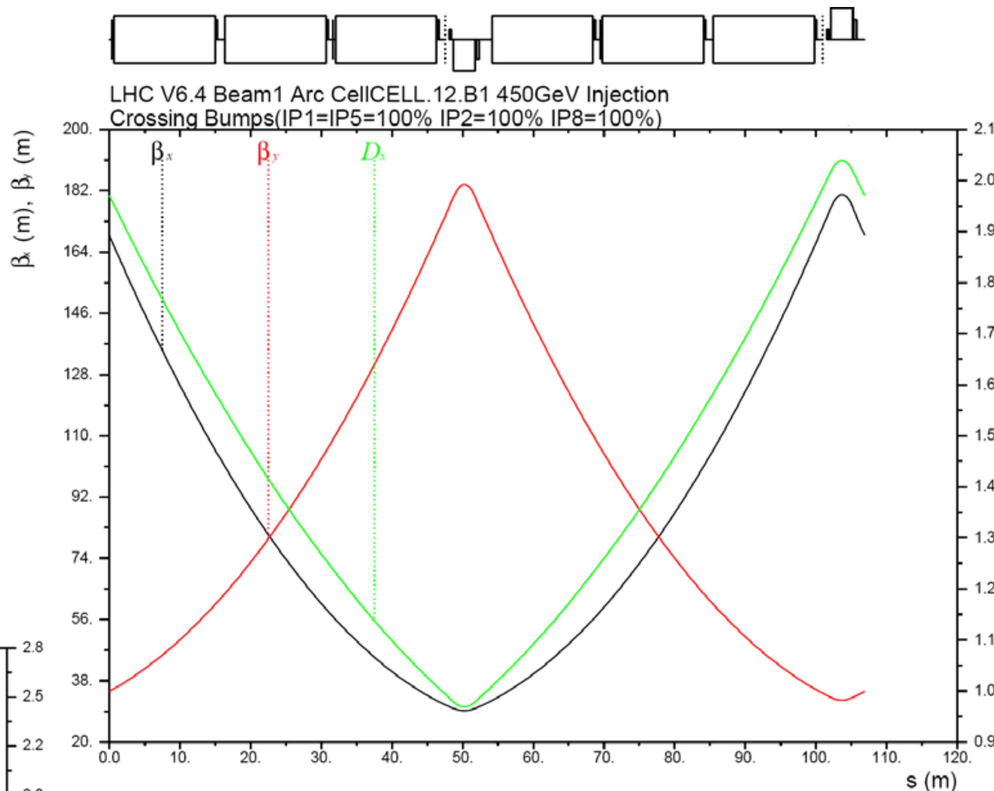
β_{ID2} , $\alpha_{ID2} = 0$, $\eta_{ID2} = \sqrt{\beta_{ID2}\,\mathcal{H}}$, $\eta'_{ID2} = 0$

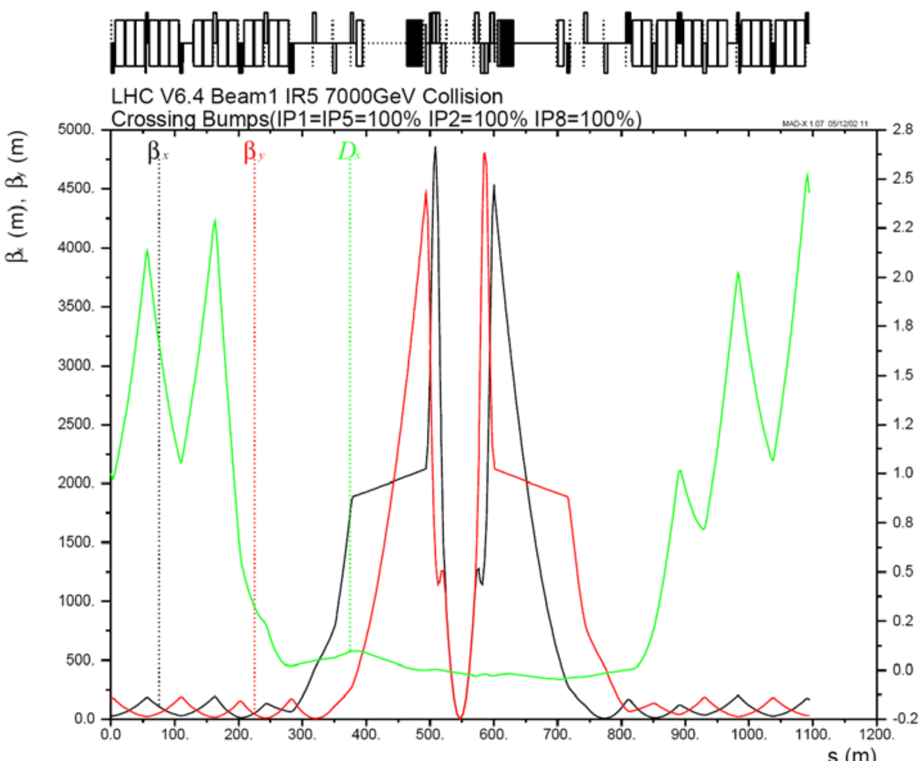
- Purpose to minimize emittance at the insertion device (increase brilliance) by imposing specific **β, α, D** and **D’** values at the entrance of the dipole
- Usually need to create achromat (dispersion equal to 0) in the straight section (**Double Bend Achromat – DBA, Triple Bend Achromat – TBA,…)**
- Try to minimize variation of beta function in the cell by tuning quadrupoles accordingly



LHC lattice examples

- FODO arc with 3+3 super-conducting bending magnets and 2 quadrupoles in between
- Beta functions between 30 and 180m





- Collision points creating beam waists with betas of **0.5m** using super-conducting quadrupoles in triplets
- Huge beta functions on triplets