Longitudinal dynamics

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Outline

- Methods of acceleration
- Energy gain and phase stability
- Momentum compaction and transition
- Equations of motion
  - Small amplitudes
  - Longitudinal invariant
- Separatrix
- Energy acceptance
- Stationary bucket
- Adiabatic damping

Electric and magnetic field

- Only electric field accelerates particles

$$\frac{dE}{dt} = \mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\mathbf{v} \cdot \mathbf{E}$$

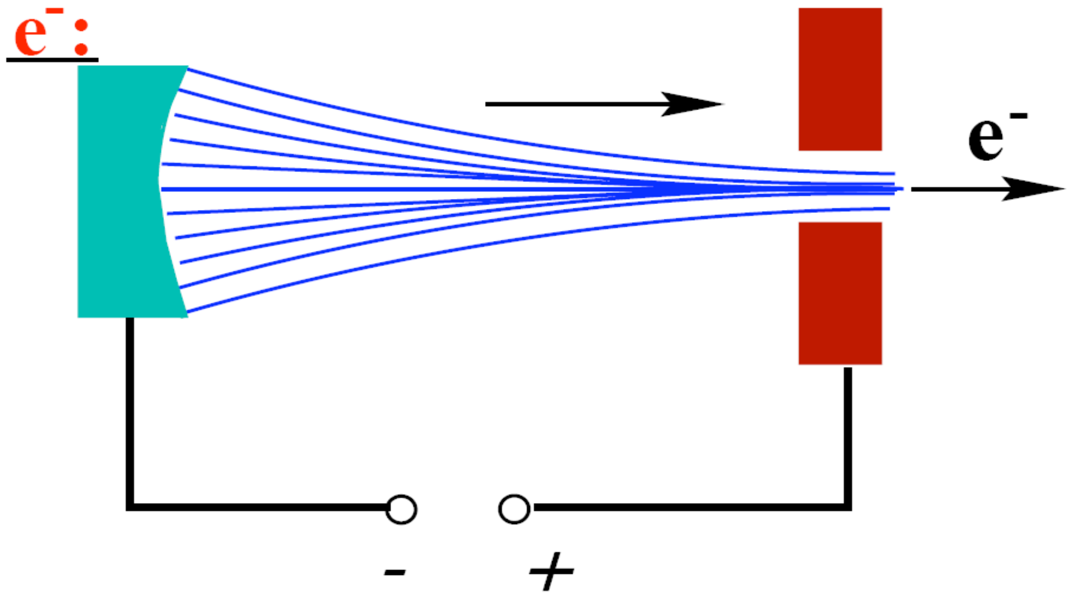
- Lorentz equation for **x** deviation of particle moving along **z** direction

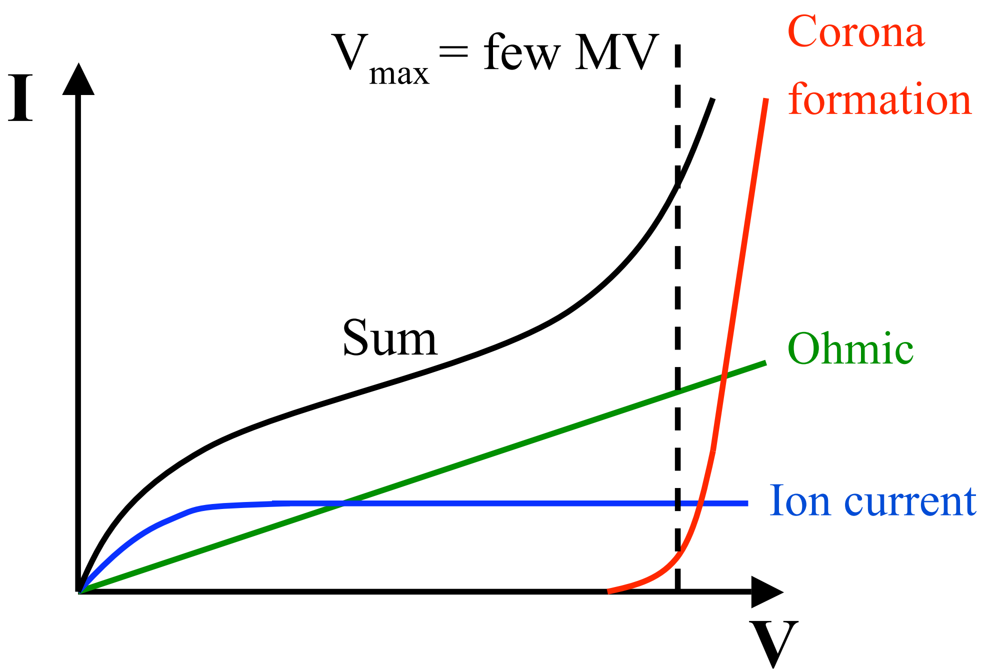
$$\frac{dp_x}{dt} = \mathbf{F_x} = q(E_x - v_z B_y)$$

- In order to have no acceleration $E_x = v_z B_y$
- For relativistic particles $E_x >> B_y$
- Magnetic field is used for guiding particles (except special cases in very low energies)

Direct voltage accelerator

- The simplest accelerator (vacuum tubes, monitors…)
  - Particle source in blue electrod
  - Accelerated by electric field in good vacuum
  - Particle exit in red electrode
  - Particle energies proportional t maximum voltage and thus limited.
  - Current components
    - Ohmic
      - Proportional to voltage
    - Residual ion current
      - Saturates rapidly
    - Corona
      - Negligible for small voltages
      - Current grows exponentially for high voltages causing spark discharge and voltage breakdown

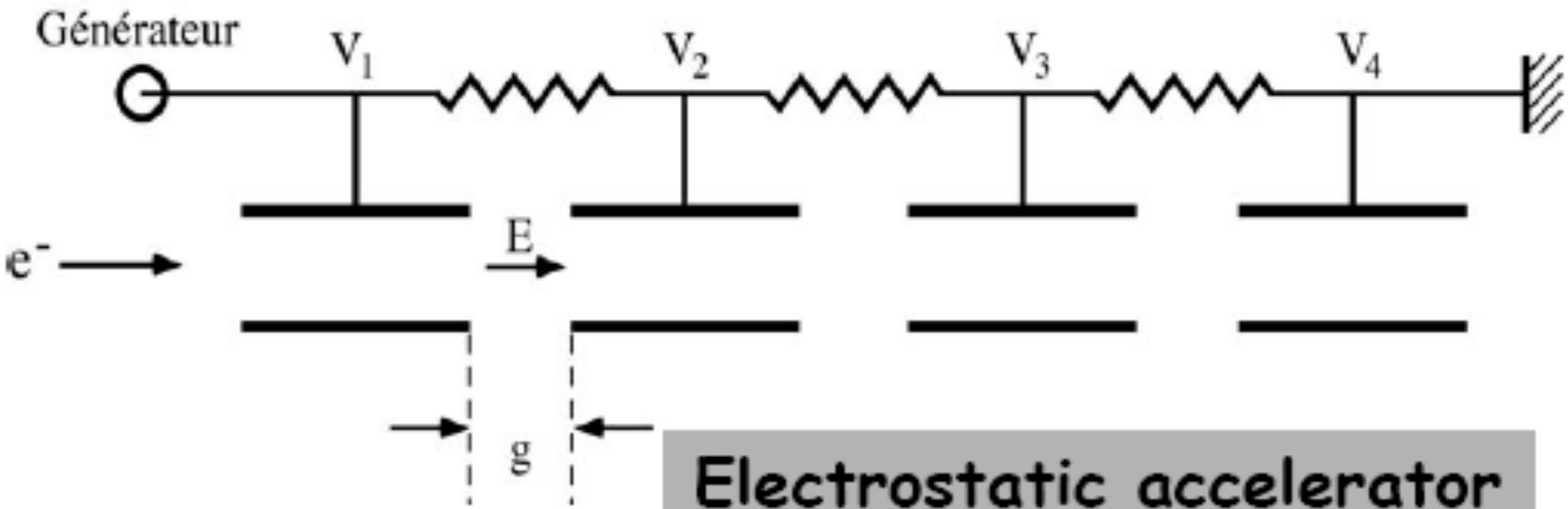




Electrostatic accelerators

- Particles are accelerated by a constant voltage across a gap
- This acceleration is limited by breakdown voltages even in the tandem or Van der Graff accelerators
- The energy gain is proportional to the generator voltage, which becomes the main limitation

$$W = nq\sum V_n = nqV_{gen}$$

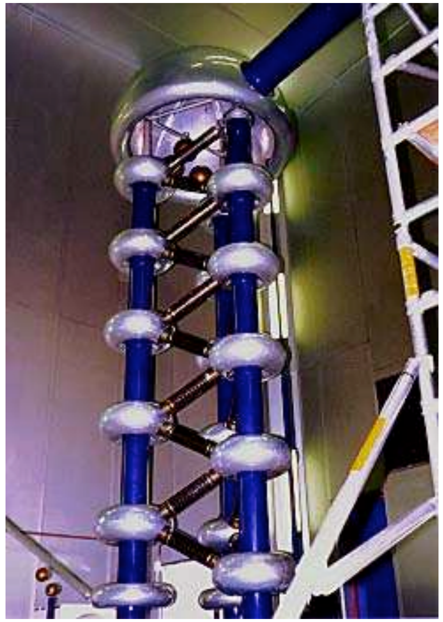


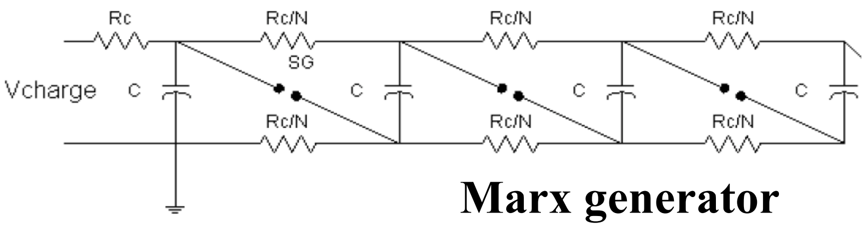
Electrostatic accelerator

Cascade generator

- Problem: generate high voltages to reach high energy
- **Cockcroft** and **Walton** (1932) developed generator based on multiple rectifiers
- Greinacker circuit operating principle
  - Transformer with sinusoidal voltage $V = V_0 \sin(\omega t)$
  - 2N diodes (current flows in one direction) ensure that max voltage in every 2 capacitors is $2V_0$, $4V_0$, $6V_0$,…,$2NV_0$
  - Reach voltage of about 4 MV but only pulsed beam currents of several hundreds of mA
- Cockcroft and Walton used accelerator to bombard Li with protons and produce an atomic reaction, giving two He nuclei (Nobel Prize 1951)
- Marx generator (1932) consists of series of resistors and capacitors, powered by high voltage $\mathbf{V}_{\mathbf{charge}}$
  - When spark discharge occurs, and as the resistance is very large, N capacitors powered in series giving total voltage of $\mathbf{NV}_{\mathbf{charge}}$

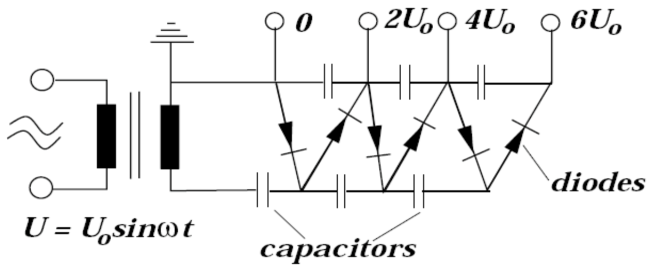
Fermilab cascade generator



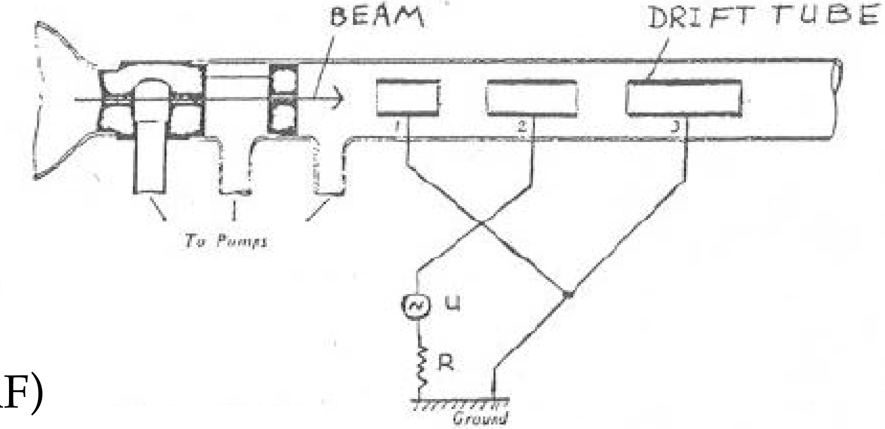


Marx generator

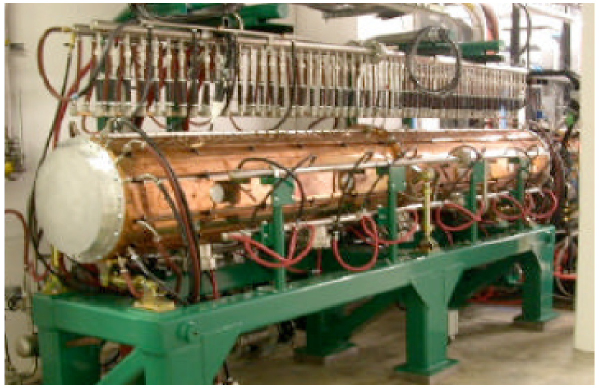
Greinacker circuit

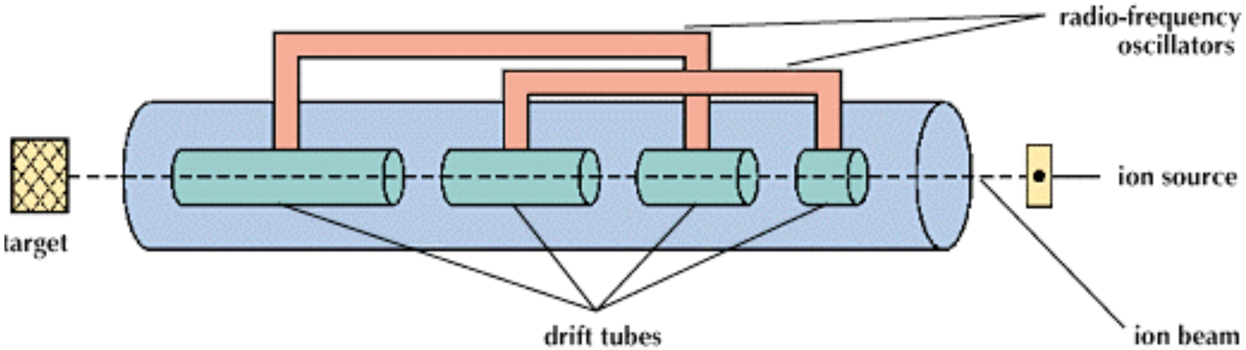


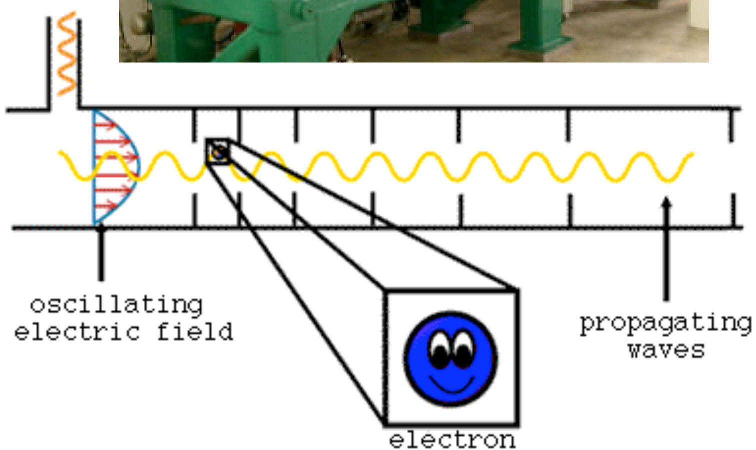
Linear accelerators



- Original idea by **Ising** (1924), first working linac by **Wideröe** (1928) and high energy (1.3 MeV) linac by **Sloan** and **Lawrence** (1931)
- Series of drift tubes alternately connected to high (RF) frequency voltage oscillator
- Particles get accelerated in gap, no effect inside tube (act like Faraday cage)
- Field reversed and then exit tube to be reaccelerated until they reach energy $E_n = nqV_0 \sin(\Psi_0)$
- For constant RF frequency, drift tubes' length increases with velocity up to relativistic limit (electrons)
- Synchronization of particle and RF field assured by **phase focusing**
- **Beams** (1933) developed first cavity structure linac (waveguides). **Hansen** and **Varian** brothers (1937) developed first klystron (frequencies up to 10GHz).
- **Alvarez** (1946) developed first DTL resonant cavity structure for protons and heavy ions



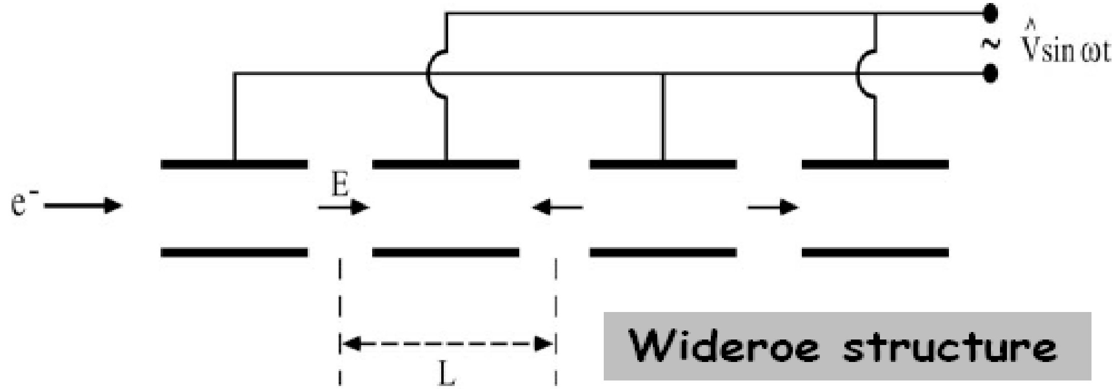


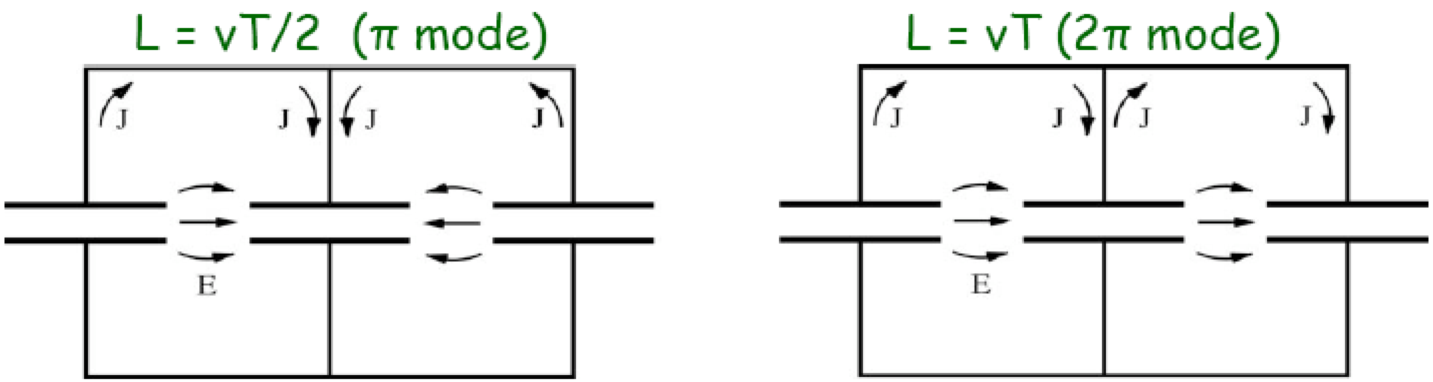


RF acceleration

- The use of RF fields allows an arbitrary number of accelerating steps in gaps and electrodes fed by RF generator
- The electric field is not longer continuous but sinusoidal alternating half periods of acceleration and deceleration
- The synchronism condition for RF period T_{RF} and particle velocity v

$$L = vT_{RF}/2 = \beta c \frac{\pi}{\omega_{RF}} = \beta\lambda/2$$





Energy gain

Assuming a sinusoidal electric field $E_z = E_0 \cos(\omega_{RF} t + \phi_s)$ where the synchronous particle passes at the middle of the gap *g*, at time *t* = *0*, the energy is

$$W(r,t) = q \int E_z dz = q \int_{-g/2}^{g/2} E_0 \cos(\omega_{RF} \frac{z}{v} + \phi_s) dz$$

And the energy gain is $\Delta W = q E_0 \int_{-q/2}^{g/2} \cos(\omega_{RF} \frac{z}{v}) dz$

and finally $\Delta W = qV \frac{\sin \Theta / 2}{\Theta / 2} = qV\,\mathrm{T}$ with the transit time factor defined as

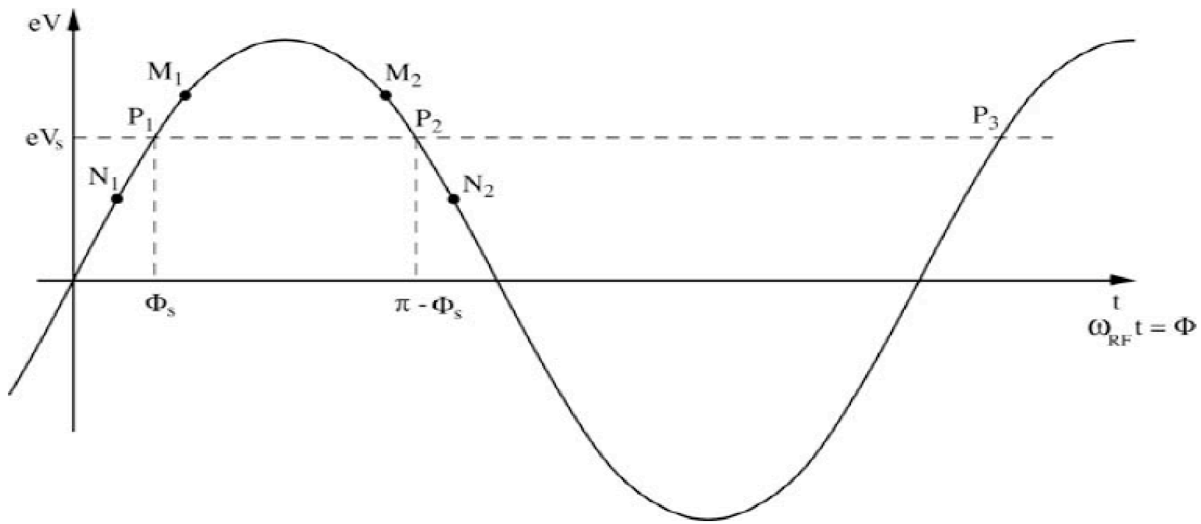
$$T = \frac{\sin(\omega g / 2v)}{\omega g / 2v}$$

It can be shown that in general

$$\mathrm{T} = \frac{\int_{-g/2}^{g/2} E(0,z) \cos \omega t(z) dz}{\int_{-g/2}^{g/2} E(0,z) dz}$$

Phase stability

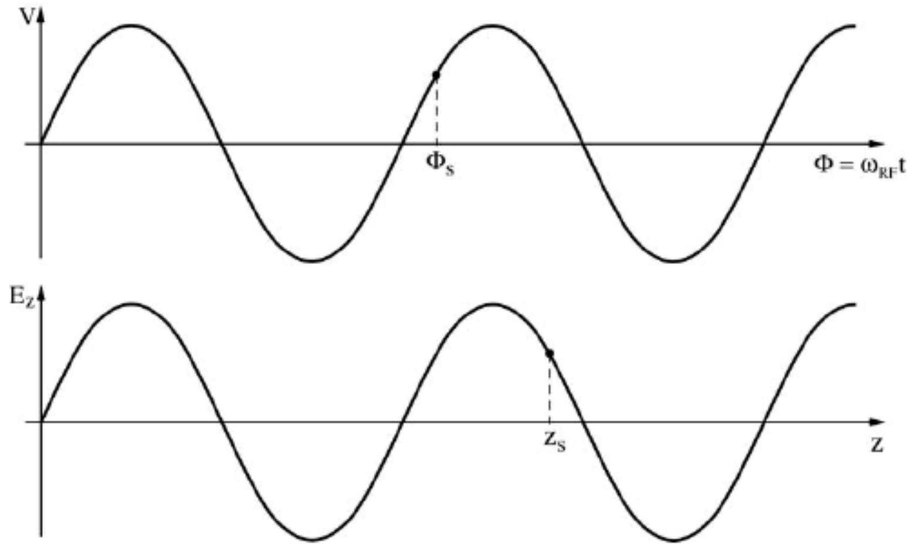
- Assume that a synchronicity condition is fulfilled at the phase ϕ_s and that energy increase produces a velocity increase
- Around point P_1, that arrives earlier (N_1) experiences a smaller accelerating field and slows down
- Particles arriving later (M1) will be accelerated more
- A restoring force that keeps particles oscillating around a stable phase called the synchronous phase ϕ_s
- The opposite happens around point P2 at π-$\phi_{s,}$ i.e. M2 and N2 will further separate



RF de-focusing

In order to have stability, the time derivative of the Voltage and the spatial derivative of the electric field should satisfy

$$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E}{\partial z} < 0$$



In the absence of electric charge the divergence of the field is given by Maxwell's equations

$$\nabla \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} > 0$$

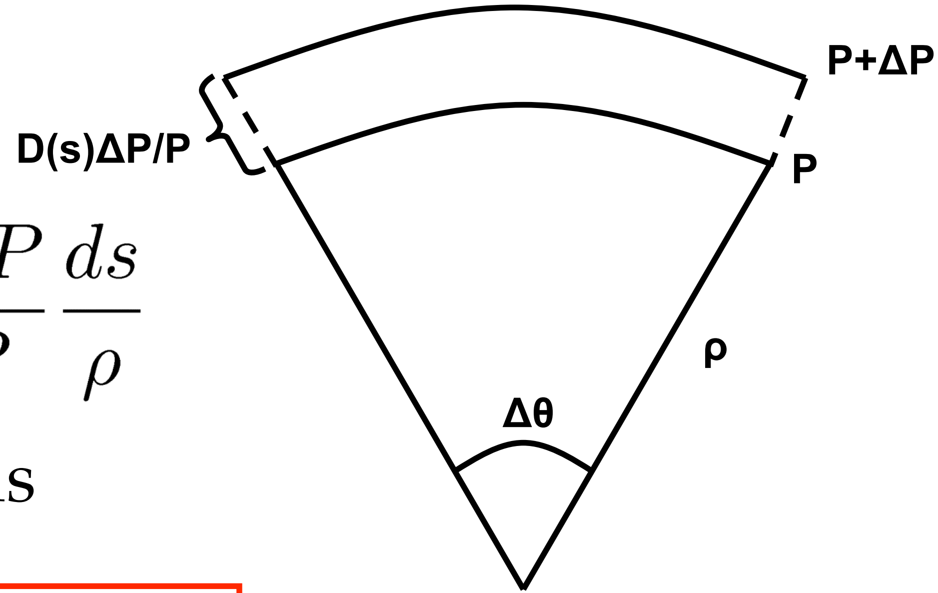
where x represents the generic transverse direction.

External focusing is required by using quadrupoles or solenoids

Momentum compaction

- Off-momentum particles on the dispersion orbit travel in a different path length than on-momentum particles
- The change of the path length with respect to the momentum spread is called **momentum compaction**

$$\alpha_c = \frac{\Delta C}{C} \Big/ \frac{\Delta P}{P}$$



- The change of circumference is

$$\Delta C = \oint D \frac{\Delta P}{P} d\theta = \oint D \frac{\Delta P}{P} \frac{ds}{\rho}$$

- So the momentum compaction is

$$\alpha_c = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds = \left\langle \frac{D(s)}{\rho(s)} \right\rangle$$

Transition energy

- The revolution frequency of a particle is $f = \frac{v}{2\pi\rho} = \frac{\beta c}{2\pi\rho}$
- The change in frequency is $\frac{\Delta f}{f} = \frac{\Delta\rho}{\rho} - \frac{\Delta\beta}{\beta}$
- From the relativistic momentum $Pc = \beta E$ we have

$$\frac{\Delta P}{P} = \frac{\Delta\beta}{\beta} + \frac{\Delta E}{E} \rightarrow \beta^2\frac{\Delta P}{P}$$ for which $\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2}\frac{\Delta P}{P}$

and the revolution frequency $\frac{\Delta f}{f} = (\frac{1}{\gamma^2} - \alpha_c)\frac{\Delta P}{P}$

The slippage factor is given by $\eta = \frac{1}{\gamma^2} - \alpha_c$

For vanishing slippage factor,
the transition energy is defined

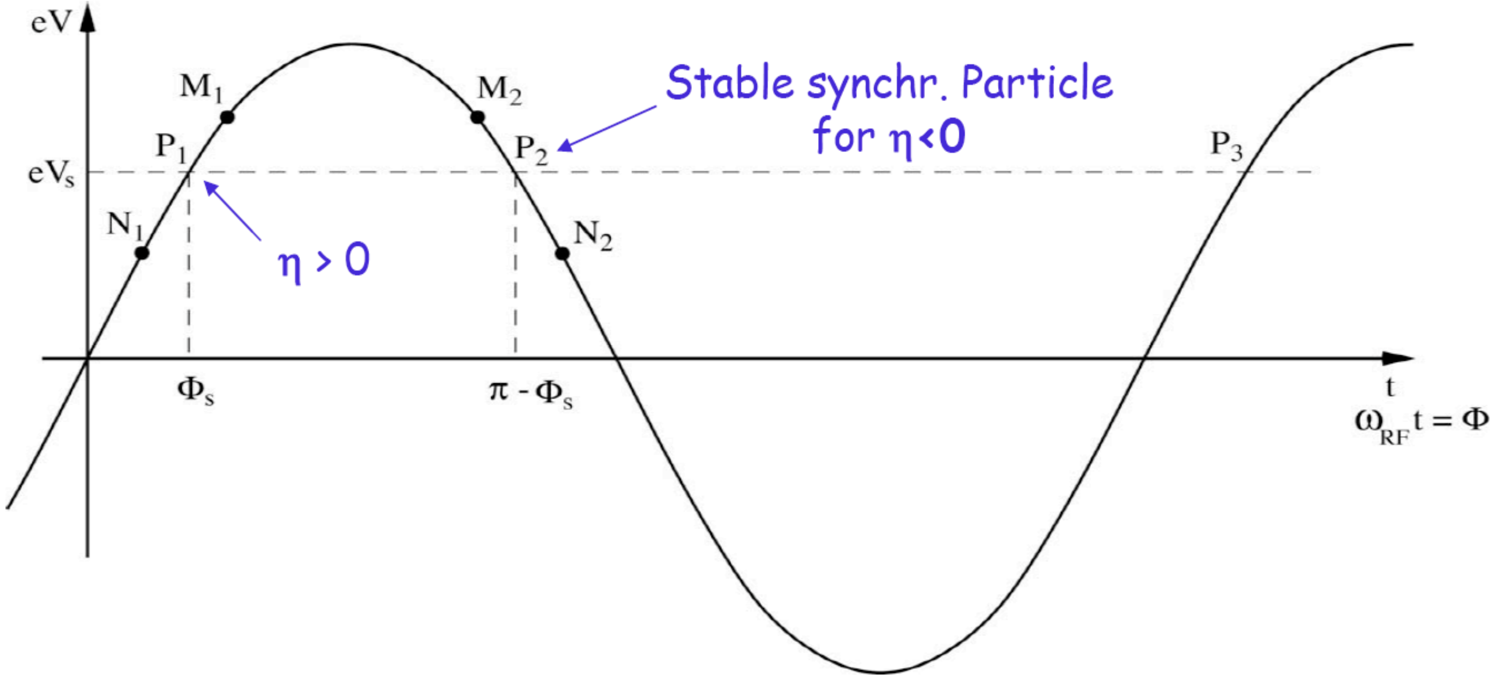
$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

Synchrotron

- Frequency modulated but also B-field increased **synchronously** to match energy and keep revolution radius constant.
- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle has the nominal energy and follow the nominal trajectory
- Magnetic field increases with momentum and the per turn change of the momentum is

$$(\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Phase stability on electron synchrotrons



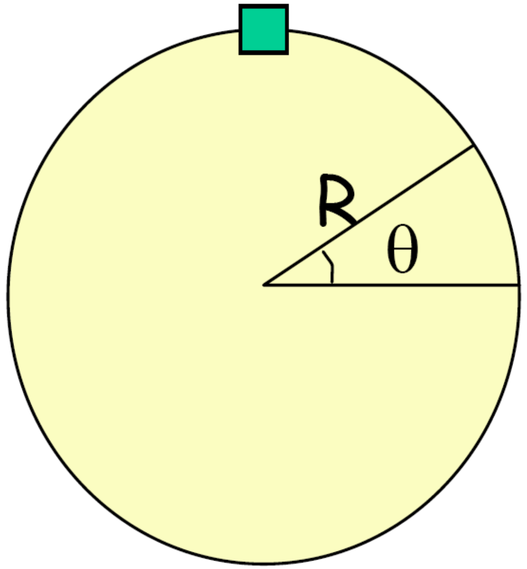
- For electron synchrotrons, the relativistic γ is very large and

$$\eta = \frac{1}{\gamma^2} - \alpha_c \approx -\alpha_c < 0$$

as momentum compaction is positive in most cases

- Above transition, an increase in energy is followed by lower revolution frequency
- A delayed particle with respect to the synchronous one will get closer to it (gets a smaller energy increase) and phase stability occurs at the point P2 (π - ϕ_s)

Energy and phase relation



- The RF frequency and phase are related to the revolution ones as follows

$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

and $\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$

- From the definition of the momentum compaction and for electrons

$$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s = \frac{E_s}{\omega_{rs}}\left(\frac{d\omega_r}{dE}\right)_s \cong -\alpha_c$$

- Replacing the revolution frequency change, the following relation is obtained between the energy and the RF phase time derivative

$$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha_c h}\frac{d\phi}{dt} = \frac{R}{c\alpha_c h}\dot{\phi}$$

Longitudinal equations of motion

- The energy gain per turn with respect to the energy gain of the synchronous particle is

$$(\Delta E)_{turn} = e\hat{V}(\sin\phi - \sin\phi_s)$$

- The rate of energy change can be approximated by

$$\frac{d(\Delta E)}{dt} \cong (\Delta E)_{turn} f_{rs} = \frac{c}{2\pi R} e\hat{V}(\sin\phi - \sin\phi_s)$$

- The second energy phase relation is written as

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s)$$

- By combining the two energy / phase relations, a 2nd order differential equation is obtained, similar the pendulum

$$\frac{d}{dt}\left(\frac{R}{c\alpha_c h}\frac{d\phi}{dt}\right) + \frac{ce\hat{V}}{2\pi R E_s}(\sin\phi - \sin\phi_s) = 0$$

Small amplitude oscillations

- Expanding the harmonic functions in the vicinity of the synchronous phase

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

- Considering also that the coefficient of the phase derivative does not change with time, the differential equation reduces to one describing an harmonic oscillator

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0 \quad \text{with frequency} \quad \Omega_s^2 = -\frac{c^2 e \alpha_c h \hat{V} \cos\phi_s}{R^2 2\pi E_s}$$

- For stability, the square of the frequency should positive and real, which gives the same relation for phase stability when particles are above transition

$$\cos\phi_s < 0 \Rightarrow \pi/2 < \phi_s < \pi$$

Longitudinal motion invariant

- For large amplitude oscillations the differential equation of the phase is written as

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

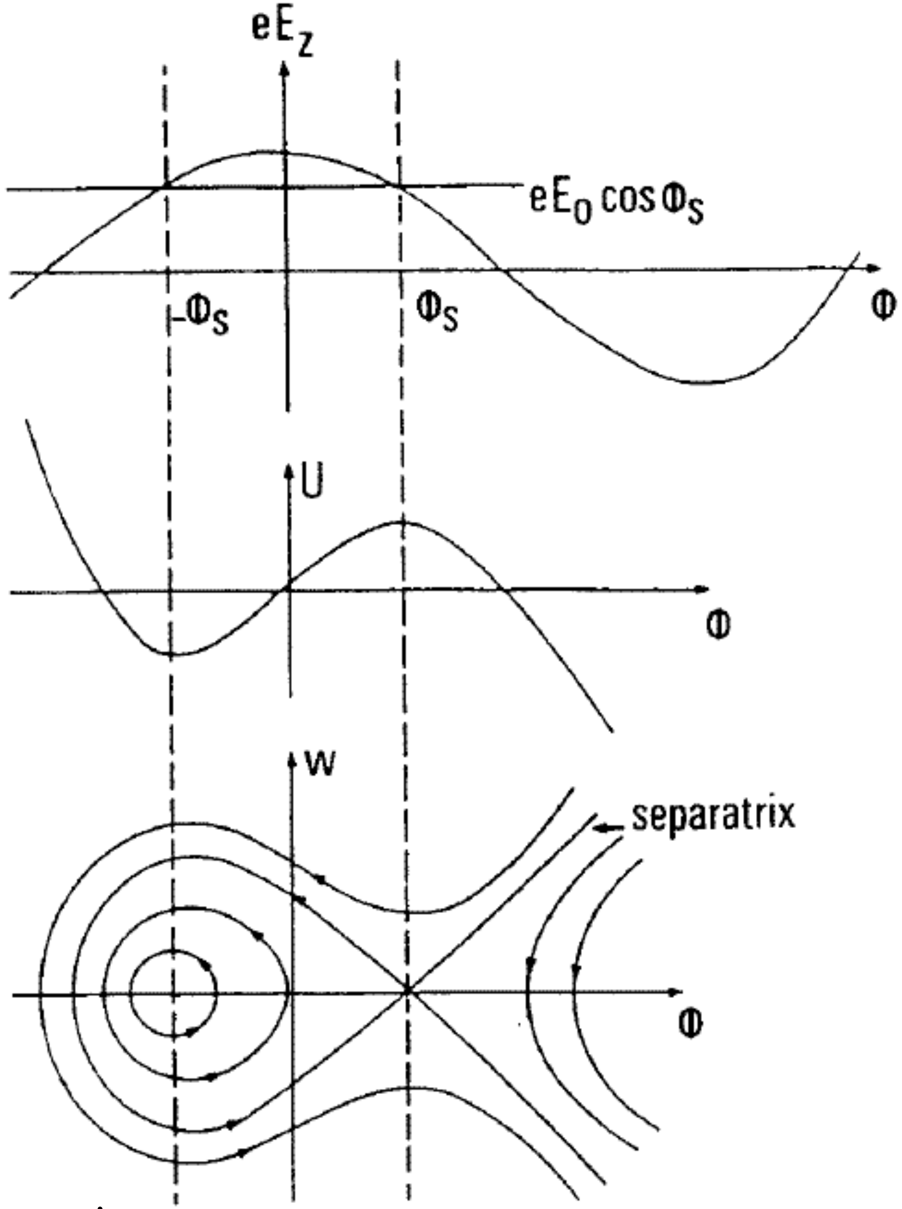
- Multiplying by the time derivative of the phase and integrating, an invariant of motion is obtained

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

reducing to the following expression, for small amplitude oscillations

$$\frac{\dot{\phi}^2}{2} + \frac{\Omega_s^2}{2}\Delta\phi = I$$

Separatrix



- In the phase space (energy change versus phase), the motion is described by distorted circles in the vicinity of ϕ_s (stable fixed point)
- For phases beyond π - ϕ_s (unstable fixed point) the motion is unbounded in the phase variable, as for the rotations of a pendulum
- The curve passing through π - ϕ_s is called the **separatrix** and the enclosed area **bucket**

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Energy acceptance

- The time derivative of the RF phase (or the energy change) reaches a maximum (the second derivative is zero) at the synchronous phase
- The equation of the separatrix at this point becomes

$$\dot{\phi}^2_{max} = 2\Omega_s^2 \left(2 + (2\phi_s - \pi) \tan\phi_s\right)$$

- Replacing the time derivative of the phase from the first energy phase relation

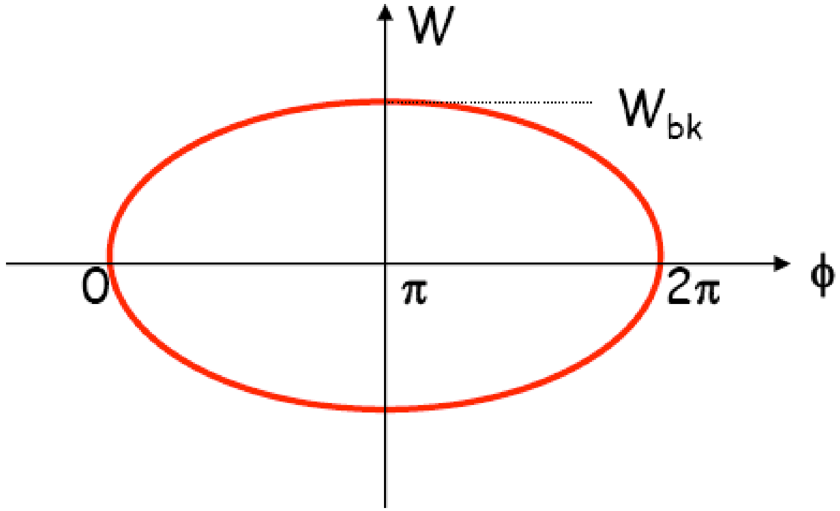
$$\left(\frac{\Delta E}{E_s}\right)_{max} = \mp \sqrt{\frac{q\hat{V}}{\pi h \alpha_c E_s} \left(2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s\right)}$$

- This equation defines the energy acceptance which depends strongly on the choice of the synchronous phase. It plays an important role on injection matching and influences strongly the electron storage ring lifetime

Stationary bucket

- When the synchronous phase is chosen to be equal to 0 (below transition) or π (above transition), there is no acceleration. The equation of the separatrix is written

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2\frac{\phi}{2}$$



- Using the (canonical) variable $W = 2\pi \frac{\Delta E}{\omega_{rs}} = 2\pi \frac{E_s R}{h\ \alpha_c\ \omega_{rs}} \dot{\phi}$ and replacing the expression for the synchrotron frequency

$$W = \pm 2\frac{C}{c}\sqrt{\frac{q\hat{V}E_s}{2\pi\ h\ \alpha_c}}\sin\frac{\phi}{2}$$

. For $\phi = \pi$, the bucket **height** is

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{e\hat{V}E_s}{2\pi\ h\ \alpha_c}}$$

and the **area** $A_{bk} = 2\int_0^{2\pi} W\ d\phi = 8W_{bk}$

Adiabatic damping

- The longitudinal oscillations can be damped directly by acceleration itself. Consider the equation of motion when the energy of the synchronous particle is not constant

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

- From this equation, we obtain a 2nd order differential equation with a damping term

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2 \Delta\phi = 0$$

- From the definition of the synchrotron frequency the damping coefficient is

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

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