Hill's equations and transport matrices

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Outline

- Hill's equations
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  - Harmonic oscillator
- Transport Matrices
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  - Dipoles
    - Sector magnets
    - Rectangular magnets
  - Doublet
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Equations of motion – Linear fields

- We ended up with the following equations

$$\left\{\begin{array}{rcl} x'' & = & \frac{1}{\rho}(1-\frac{x}{\rho}) - \frac{qB_y}{P} \\ y'' & = & \frac{qB_x}{P} \end{array}\right.$$

- Consider s-dependent fields from dipoles and normal quadrupoles $B_y = B_0(s) - G(s)x\ , \quad B_x = -G(s)y$
- The total momentum can be written $P = P_0(1+\frac{\Delta P}{P})$
- The magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and the normalized gradient $k = \frac{G}{B_0\rho}$
- The equations become

$$\begin{array}{rcl} x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right)x & = & \frac{1}{\rho(s)}\frac{\Delta P}{P} \\ y'' + k(s)\ y & = & 0 \end{array}$$

- Inhomogeneous equations with s-dependent coefficients
- Note that the term **$1/\rho^2$** corresponds to the dipole **week focusing**
- The term **$\Delta P/(P\rho)$** is present for off-momentum particles

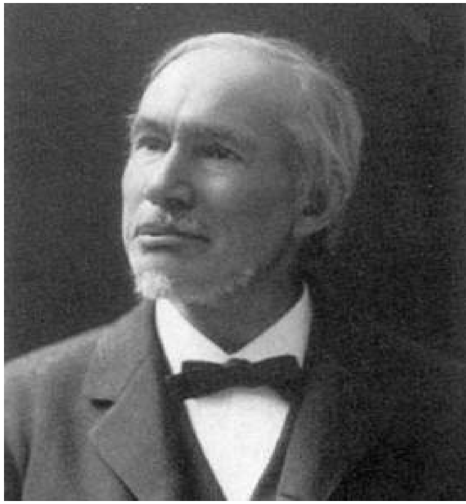
Hill's equations

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations

- Consider particles with the design momentum. The equations of motion become

$$
\begin{aligned}
x'' + K_x(s)\ x &= 0 \\
y'' + K_y(s)\ y &= 0
\end{aligned}
$$

with $K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right)$, $K_y(s) = k(s)$



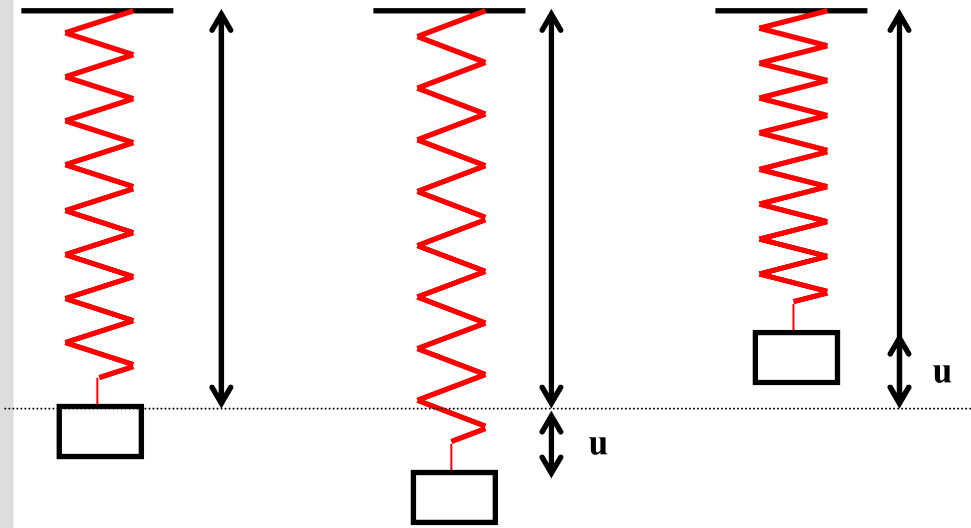
George Hill

- **Hill's equations of linear transverse particle motion**

- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)

- In a ring or in transport line with symmetries, coefficients are periodic $K_x(s) = K_x(s+C)$, $K_y(s) = K_y(s+C)$

- Not feasible to get analytical solutions for all accelerator



- Consider K(s) = k_0 = constant

$$u'' + k_0 \, u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\ u(0) + S(s)\ u'(0) \\ u'(s) &= C'(s) u(0) + S'(s) u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0} s) \ , \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0} s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|} s) \ , \ S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|} s) \quad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$

  which is always true for conservative systems

- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- The accelerator can be build by a series of matrix multiplications

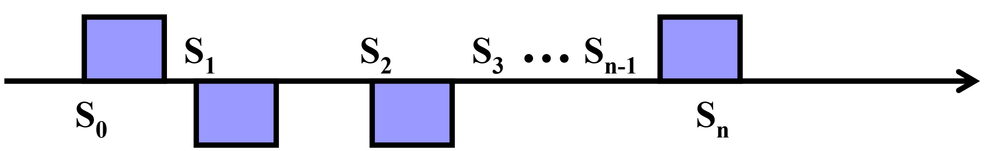
$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \underbrace{\mathcal{M}(s_1|s0)}_{\text{from } s_0 \text{ to } s_1}$$

from $\mathbf{s_0}$ to $\mathbf{s_1}$

from $\mathbf{s_0}$ to $\mathbf{s_2}$

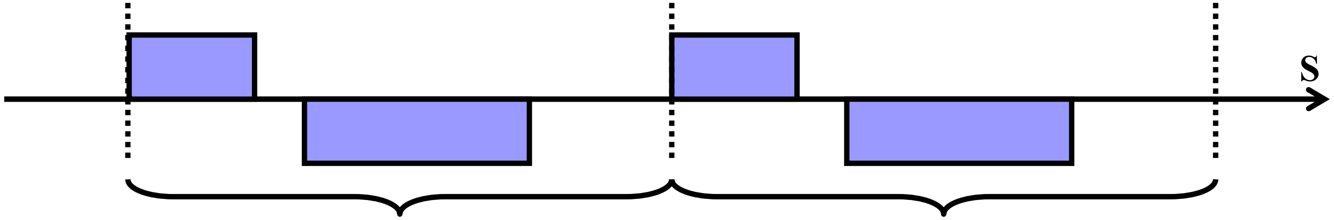
from $\mathbf{s_0}$ to $\mathbf{s_3}$

from $\mathbf{s_0}$ to $\mathbf{s_n}$



Symmetric lines

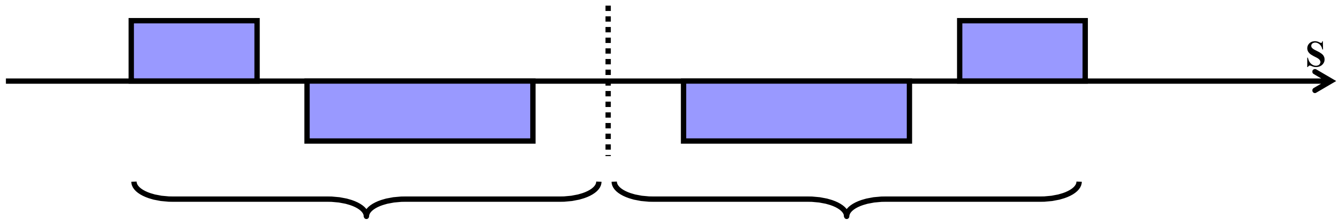
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\rm tot} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\rm tot} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Transfer matrix of a drift

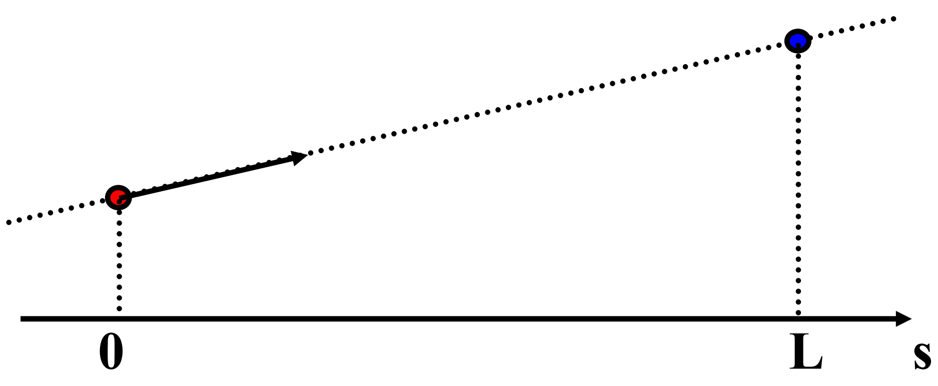
- Consider a drift (no magnetic elements) of length $L=s-s_0$

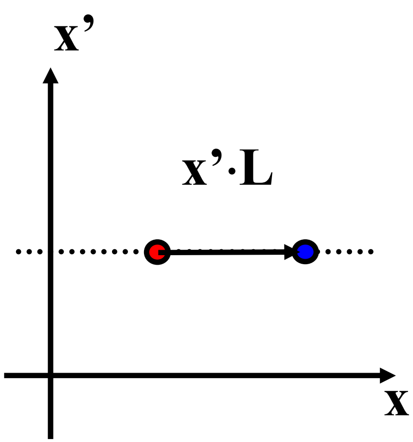
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\mathcal{M}_{\text{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} u(s) &= u_0 + (s-s_0)u'_0 = u_0 + Lu'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if there is a slope. Slope remains unchanged





Focusing - defocusing thin lens

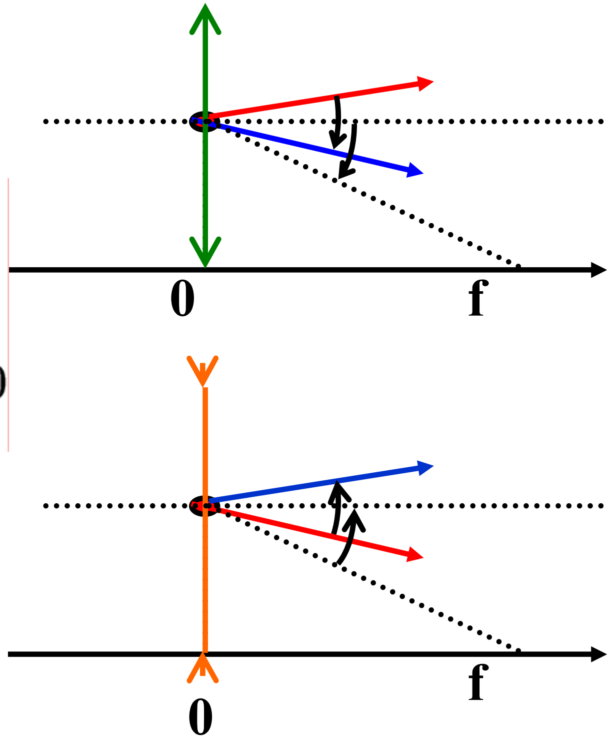
- Consider a lens with focal length **±f**

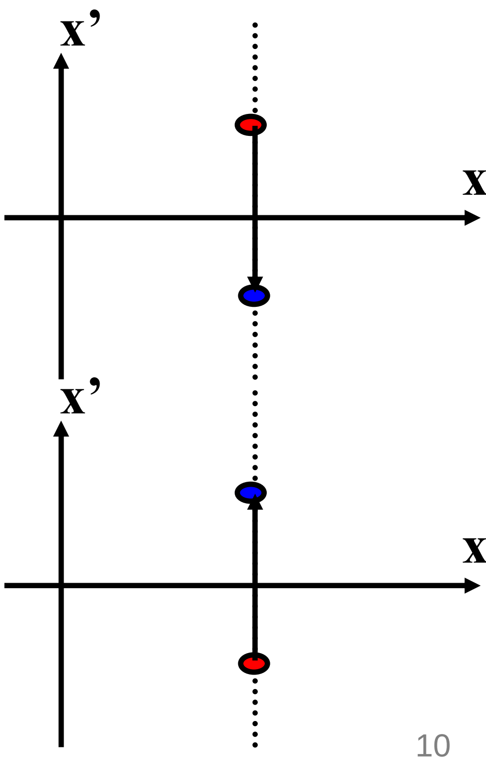
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}$$

- Slope diminishes (focusing) or increases (defocusing). Position remains unchanged

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= u_0 \mp \frac{1}{f} u'_0 \end{aligned}$$

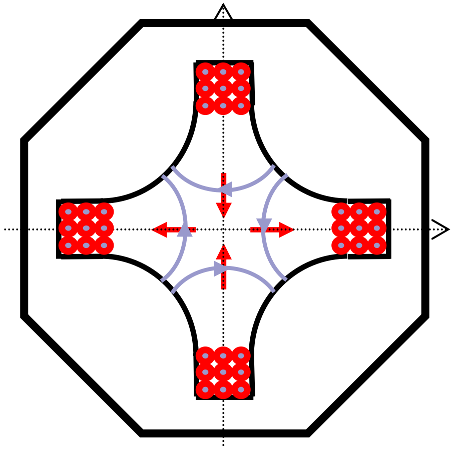




Quadrupole

- Consider a quadrupole magnet of length **L**. The field is

$$B_y = -G(s)x \ , \quad B_x = -G(s)y$$

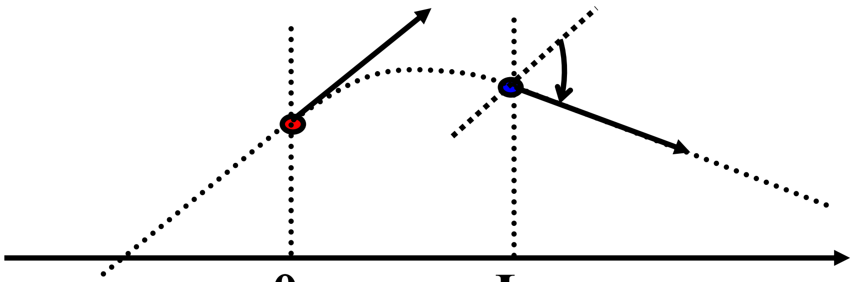


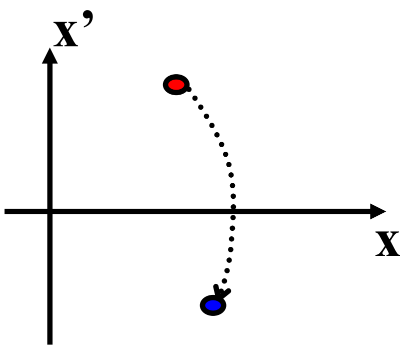
- with normalized quadrupole gradient (in **m**$^{-2}$)

$$k = \frac{G}{B_0 \rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$





Quadrupole II

- For a focusing quad (**k>0**)

$$\mathcal{M}_{\rm QF} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

- For a defocusing quad (**k<0**)

$$\mathcal{M}_{\rm QD} = \begin{pmatrix} \cosh(\sqrt{|k|}L) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}L) \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix}$$

- By setting $\sqrt{k}L \to 0$

$$\mathcal{M}_{\rm QF,QD} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \mathcal{M}_{\rm lens}$$

Sector Dipole

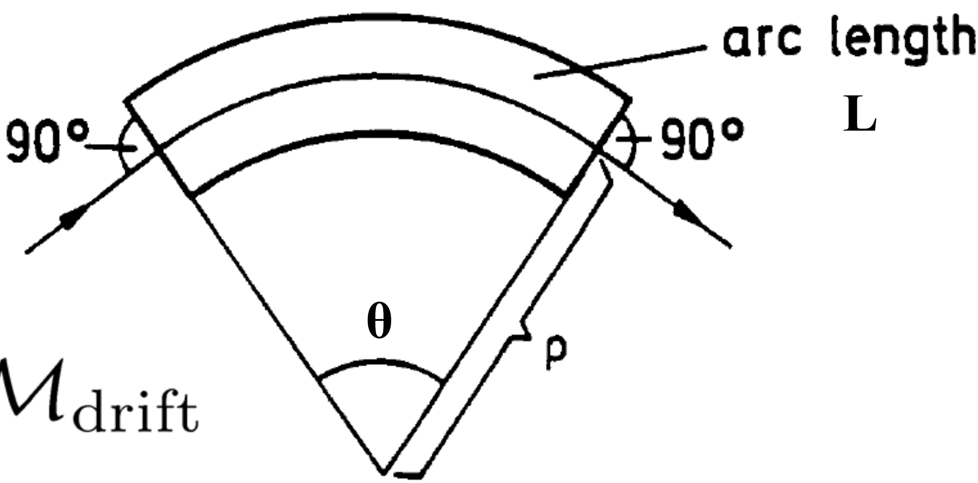
- Consider a dipole of length **L**. By setting in the focusing quadrupole matrix

$$k = \frac{1}{\rho^2} > 0$$

the transfer matrix for a sector dipole becomes

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

with a bending radius $\theta = \frac{L}{\rho}$

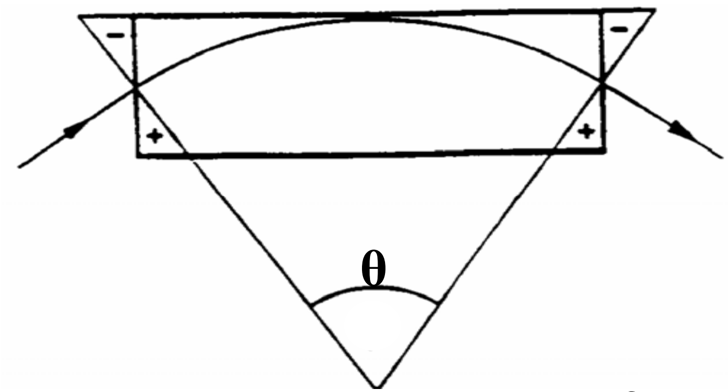


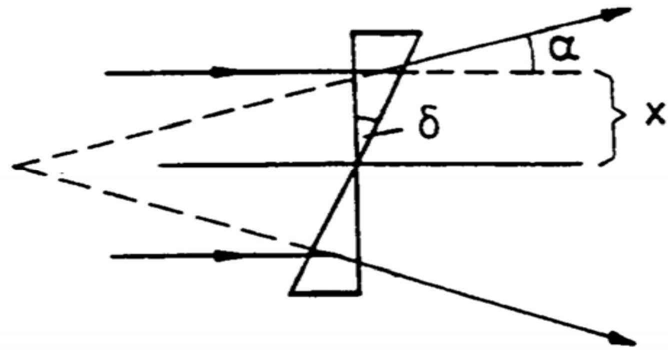
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane

Rectangular Dipole





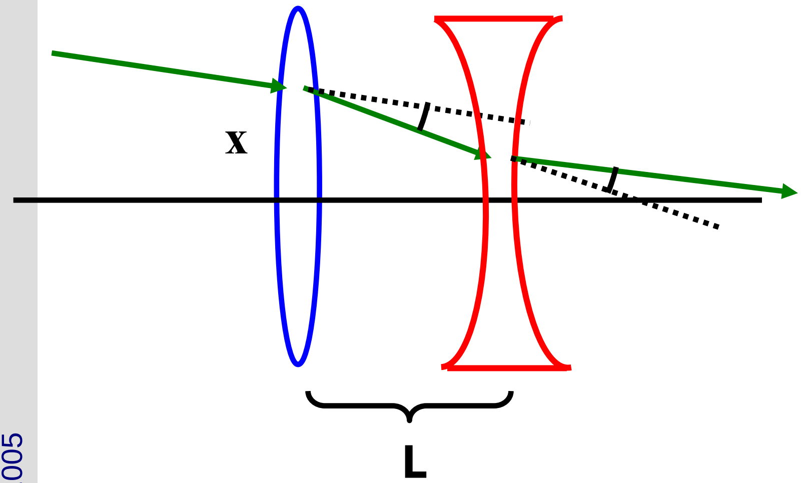
- Consider a rectangular dipole of length **L**. At each edge, the deflecting angle is

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$

It acts as a thin defocusing lens with focal length $\frac{1}{f} = \frac{\tan\delta}{\rho}$

- The transfer matrix is $\mathcal{M}_{\mathrm{rect}} = \mathcal{M}_{\mathrm{edge}} \cdot \mathcal{M}_{\mathrm{sector}} \cdot \mathcal{M}_{\mathrm{edge}}$ with $\mathcal{M}_{\mathrm{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$
- For **θ<<1**, **δ=θ/2**.
- In deflecting plane (like **drift**) in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\mathrm{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\mathrm{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

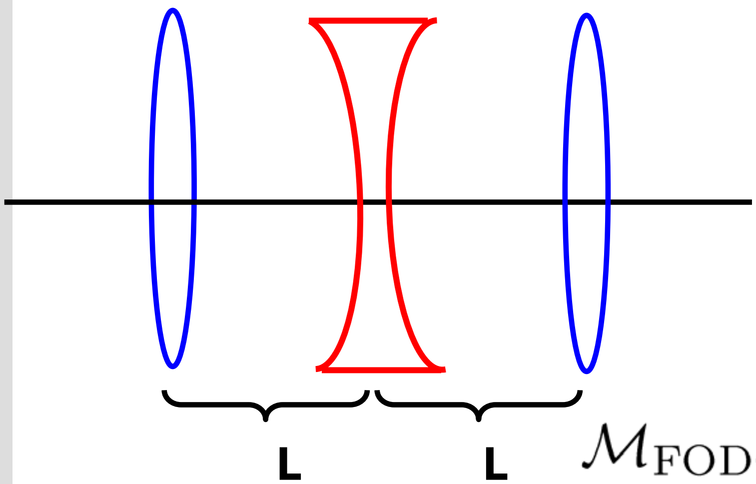


- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths $\mathbf{f_1}$ and $\mathbf{f_2}$ separated by a distance **L**.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with the **total focal length** $$\frac{1}{f^\star} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1\ f_2}$$

- Setting $\mathbf{f_1 = - f_2 = f}$ $$\frac{1}{f^\star} = \frac{L}{f^2}$$
- Alternating gradient focusing seems overall focusing
- This is only valid in thin lens approximation!!!



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- The symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

with the transfer matrices

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ \frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}$$