Non-linear dynamics

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Summary

- Driven oscillators and resonance condition
- Field imperfections and normalized field errors
- Perturbation treatment for a sextupole
- Poincaré section
- Chaotic motion
- Singe-particle diffusion
  - Dynamics aperture
  - Frequency maps

Damped and driven harmonic oscillator

- Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = 0$$

  - Q is the damping coefficient (amplitude decreases with time)
  - ω_0 is the Eigenfrequency of the harmonic oscillator
- An external force can pump energy into the system

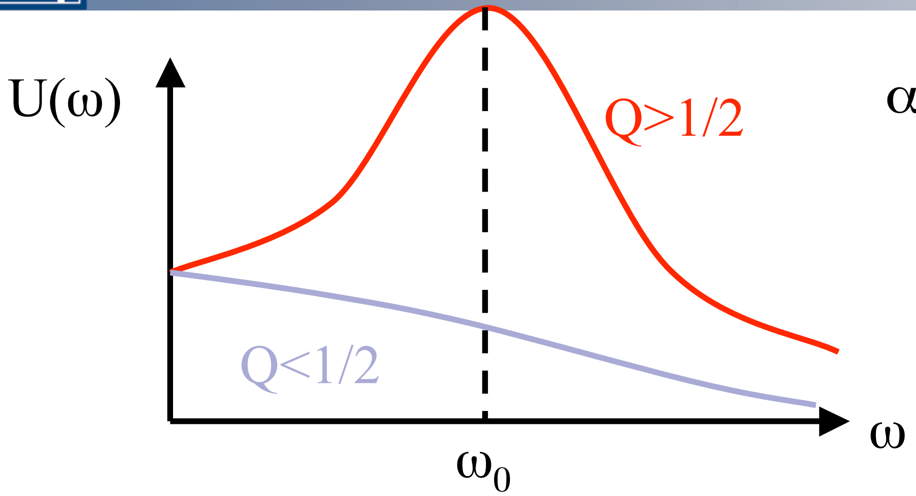
$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

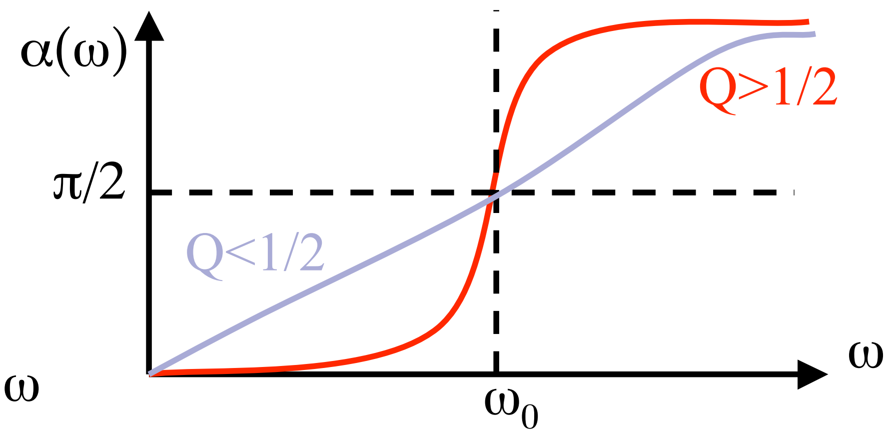
- General solution $u(t) = u_h(t) + u_{st}(t)$ with

$$u_{st}(t) = U(\omega)\cos(\omega t - a(\omega))$$

  - **ω** the frequency of the driven oscillation
  - Amplitude **U(ω)** can become large for certain frequencies

Resonance effect



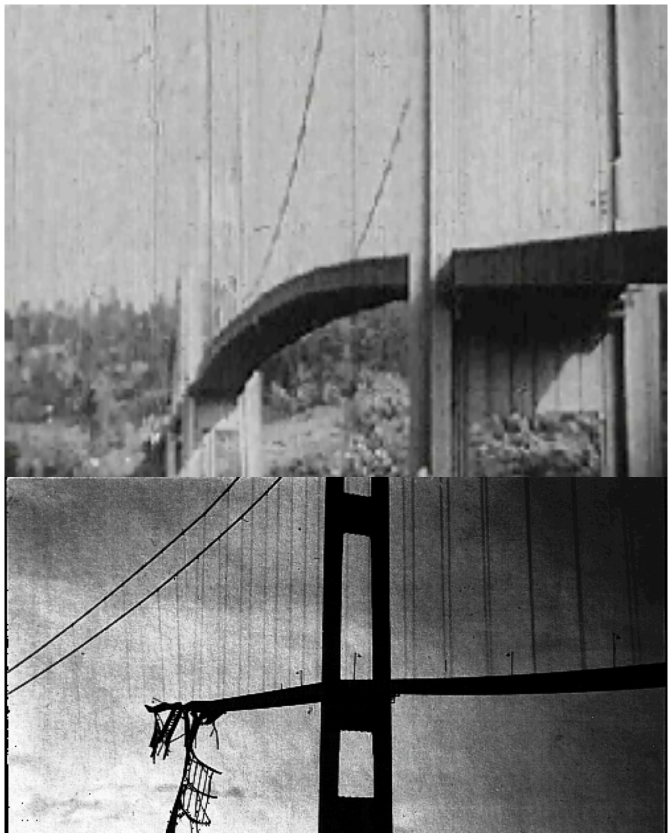


$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

excitation by strong wind on the eigenfrequencies



Perturbation in Hills equations

- Hill's equations with driven harmonic force

$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = F(u(s), s)$$

where the F is the Lorentz force from perturbing fields

  - **Linear magnet imperfections**: derivation from the design dipole and quadrupole fields due to powering and alignment errors
  - **Time varying fields**: feedback systems (damper) and wake fields due to collective effects (wall currents)
  - **Non-linear magnets**: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
  - **Beam-beam interactions**: strongly non-linear field
  - **Space charge effects**: very important for high intensity beams
  - **non-linear magnetic field imperfections**: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

Localized Perturbation

- Periodic delta function

$$\delta_L(s-s_0) = \begin{cases} 1 & \text{for `s' = } s_0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \oint \delta_L(s-s_0)ds = 1$$

- Equation of motion for a single perturbation in the storage ring

$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = \delta_L(s-s_0)lF(u(s),s)$$

- Expanding in Fourier series the delta function

$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = \frac{l}{C}\sum_m \cos(2\pi m\frac{s}{C})F(u(s),s)$$

- Infinite number of driving frequencies!!!
- Recall that the driving force can be the result of any multi-pole

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

Resonance conditions and tune diagram

- Equations of motion (u = x or y) including all multi-pole errors

$$\frac{d^2u(s)}{ds^2} + \omega_0^2 u(s) = \epsilon \sum_{n_x+n_y<n,m} a_{n_x,n_y,m} x^{n_x} y^{n_y} \cos(2\pi m \frac{s}{C})$$

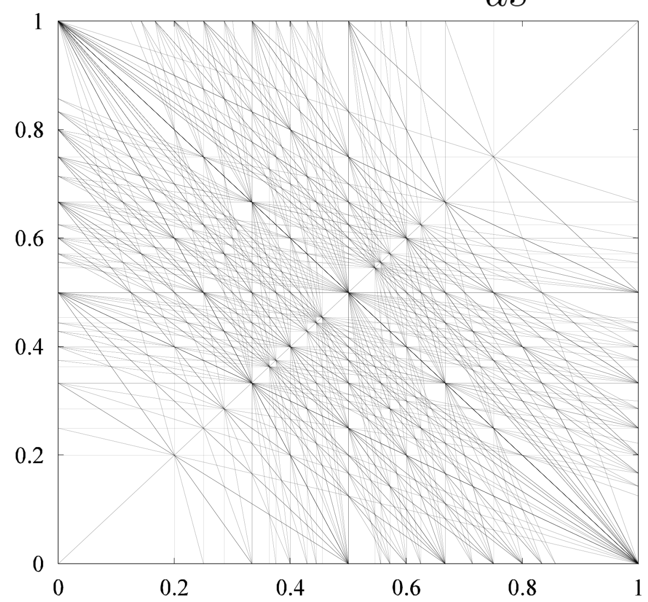
- Solved with perturbation theory approach

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots \qquad \text{with}$$

$$u_0(s) = u_0 \cos(2\pi Q s/C + \phi_0) \ \text{ and } \ \omega_0 = \frac{2\pi}{C} Q$$

- At first order

$$\frac{d^2u_1(s)}{ds^2} + \omega_0^2 u_1(s) = \epsilon \sum_{n'_x<n_x, n'_y<n_y} \bar{a}_{n'_x,n'_y,m} \cos(\frac{2\pi s}{C}(n'_x Q_x + n'_y Q_y + m))$$



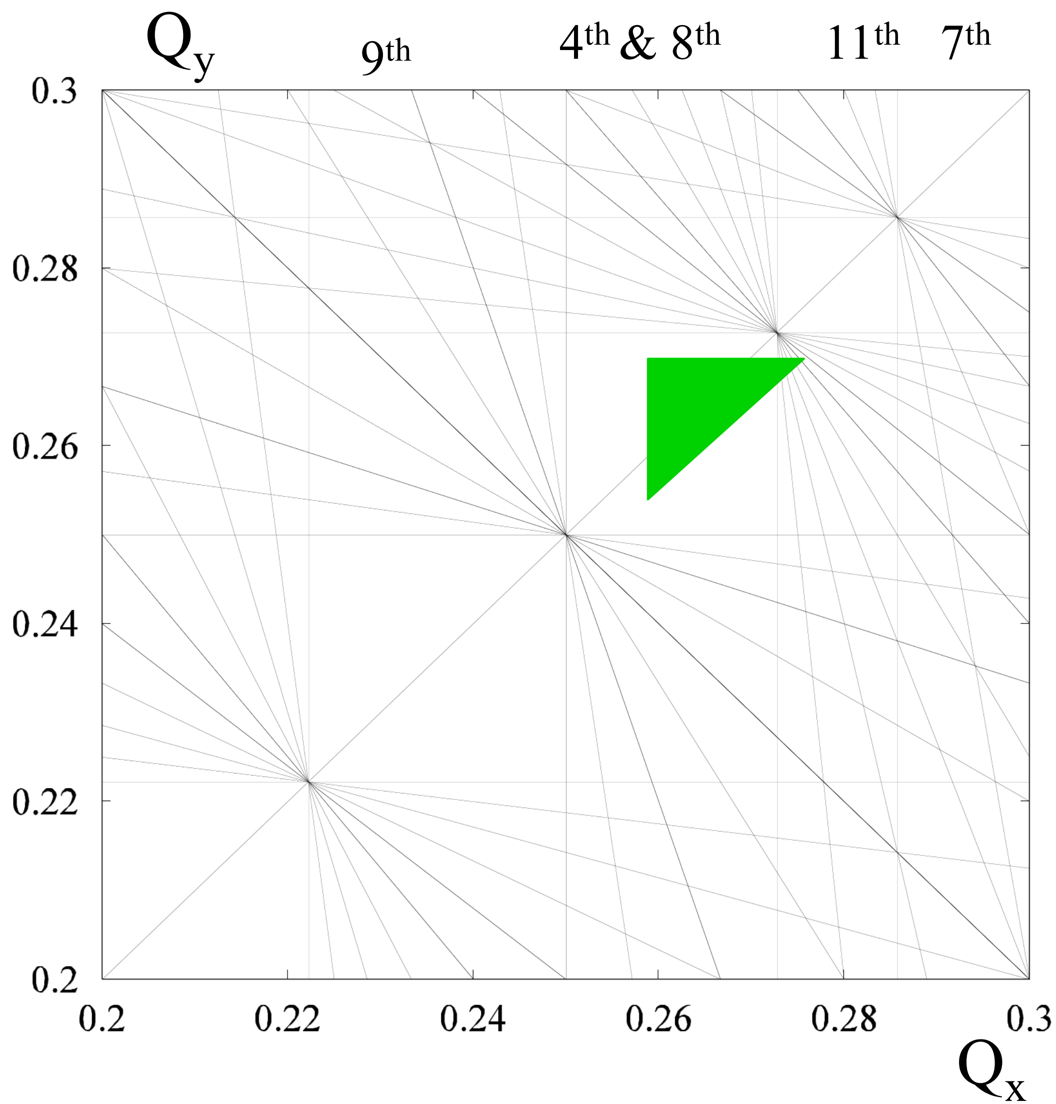
Resonance condition

$$n'_x Q_x + n'_y Q_y + m = 0$$

There are resonance lines everywhere !!!

Choice of the working point

- Regions with few resonances:
  $n'_x Q_x + n'_y Q_y + m = 0$
- Avoid low order resonances
- < 12th order for a proton beam without damping
- < 3rd ⇔ 5th order for electron beams with damping
- Close to coupling resonances: regions without low order resonances but relatively small!



Single Sextupole Perturbation

- Consider a thin sextupole perturbation $F(S) = \delta(s - s_0) l \frac{S}{2} x_0^2$
- Equations of motion

$$\frac{d^2 x_1(s)}{ds^2} + \omega_0^2 x_1(s) = x_0^2 \frac{Sl}{2C} \sum_m \cos(2\pi m \frac{s}{C})$$

- With $x_0(s) = x_0 \cos(\omega_0 s + \phi_0)$
- The equation is written

$$\frac{d^2 x_1(s)}{ds^2} + \omega_0^2 x_1(s) = \frac{Sl}{2C} A^2 \sum_m \cos(2\pi m \frac{s}{C}) + \frac{Sl}{8C} A^2 \sum_m \cos(2\pi (m \pm 2Q_x) \frac{s}{C})$$

- Resonance conditions $\underbrace{Q_x + m = 0}_{\text{integer}}$ $\underbrace{3Q_x + m = 0}_{\text{third integer}}$
- No exact solution
- Need numerical tools to integrate equations of motion

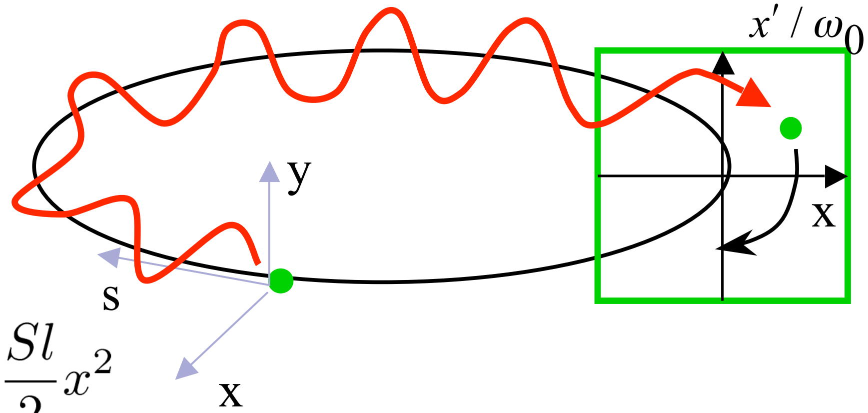
Poincaré Section

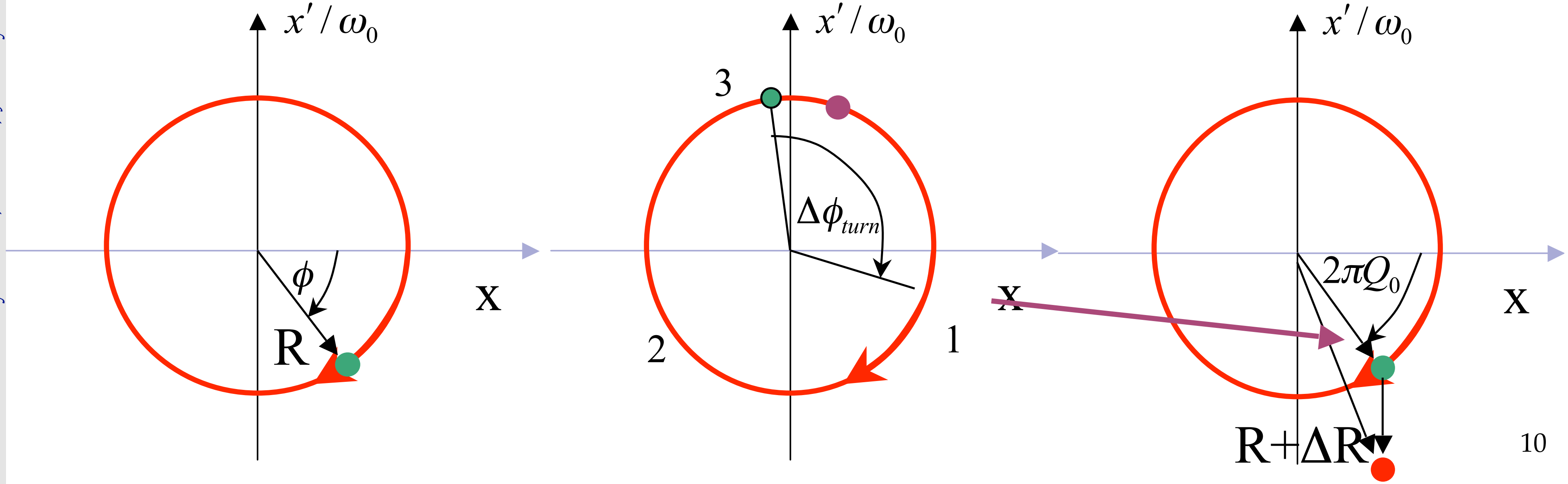
- Record the particle coordinates at one location (BPM)
- Unperturbed motion lies on a circle (simple rotation)
- Resonance is described by fixed points
- For a sextupole

$$\Delta x' = \oint F(s)ds \rightarrow \Delta x' = \frac{Sl}{2}x^2$$

- The particle does not lie on a circle!
- The change of tune per turn is $\Delta Q \propto x^2$

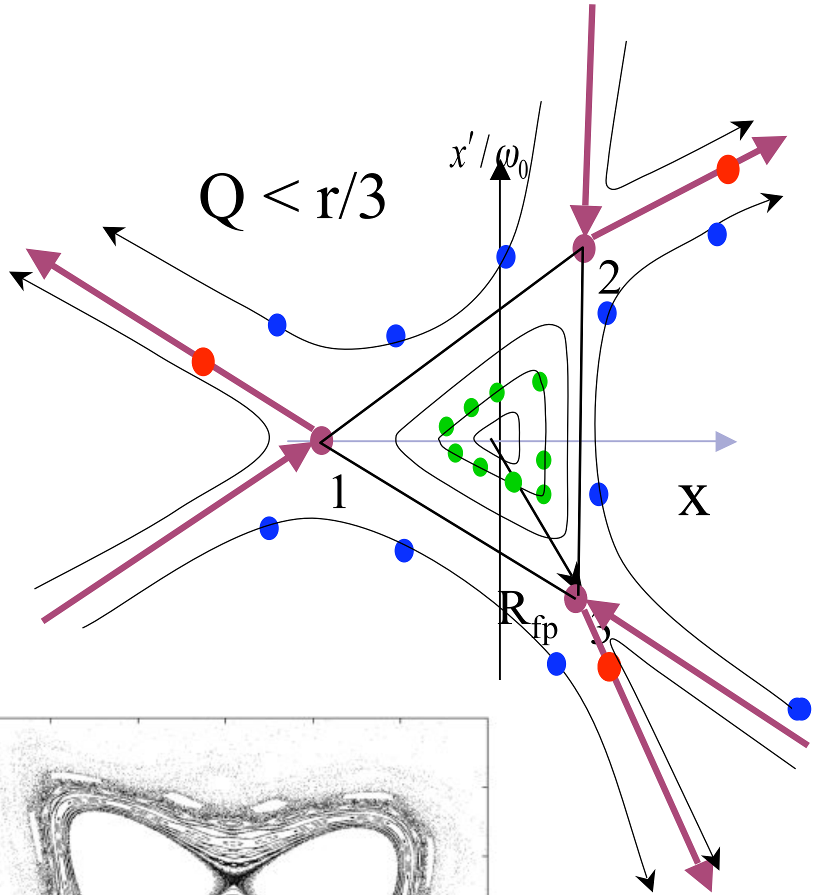
Poincaré Section:

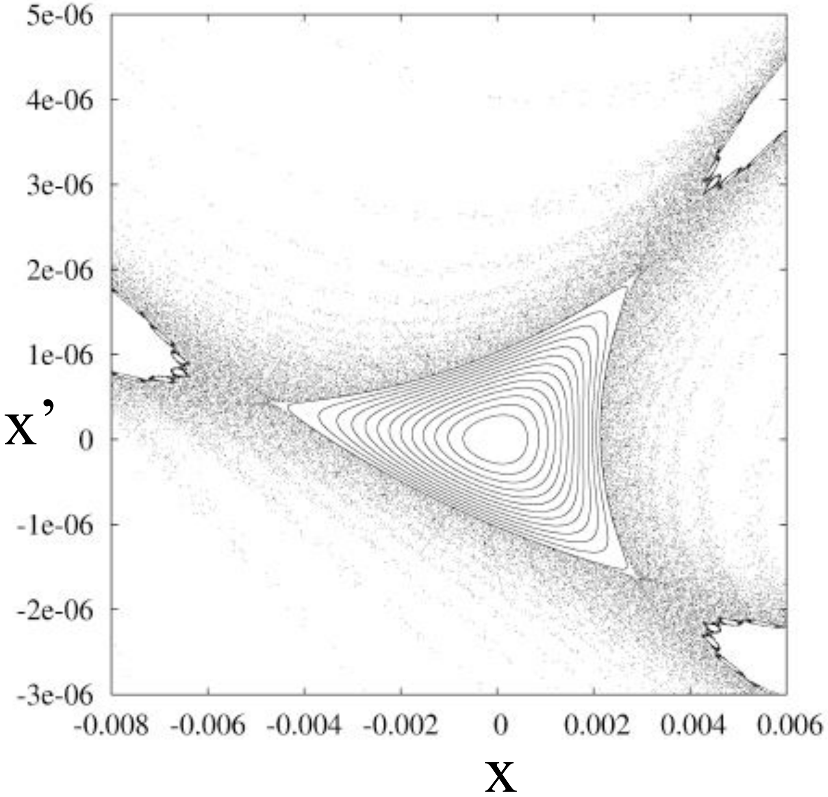


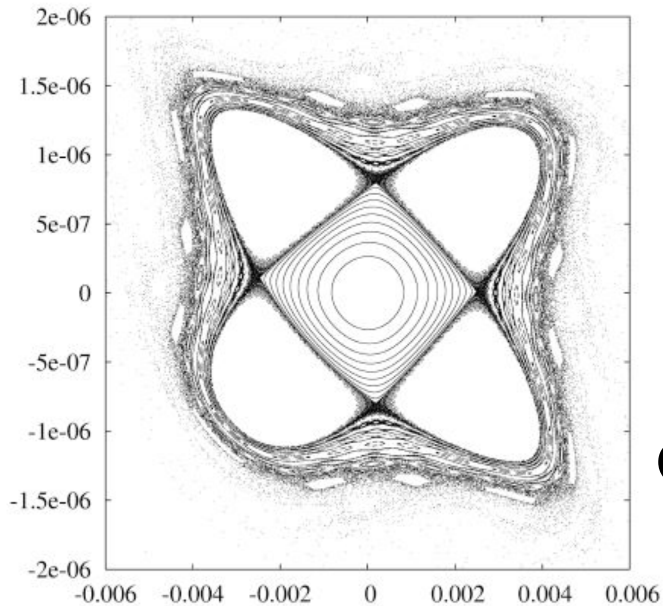


Topology of a sextupole resonance

- Small amplitude, regular motion
- Large amplitude, instability, chaotic motion and particle loss
- Separatrix: barrier between stable and unstable motion (location of unstable fixed points)







Octupole

Sextupole effects

9 first order terms:

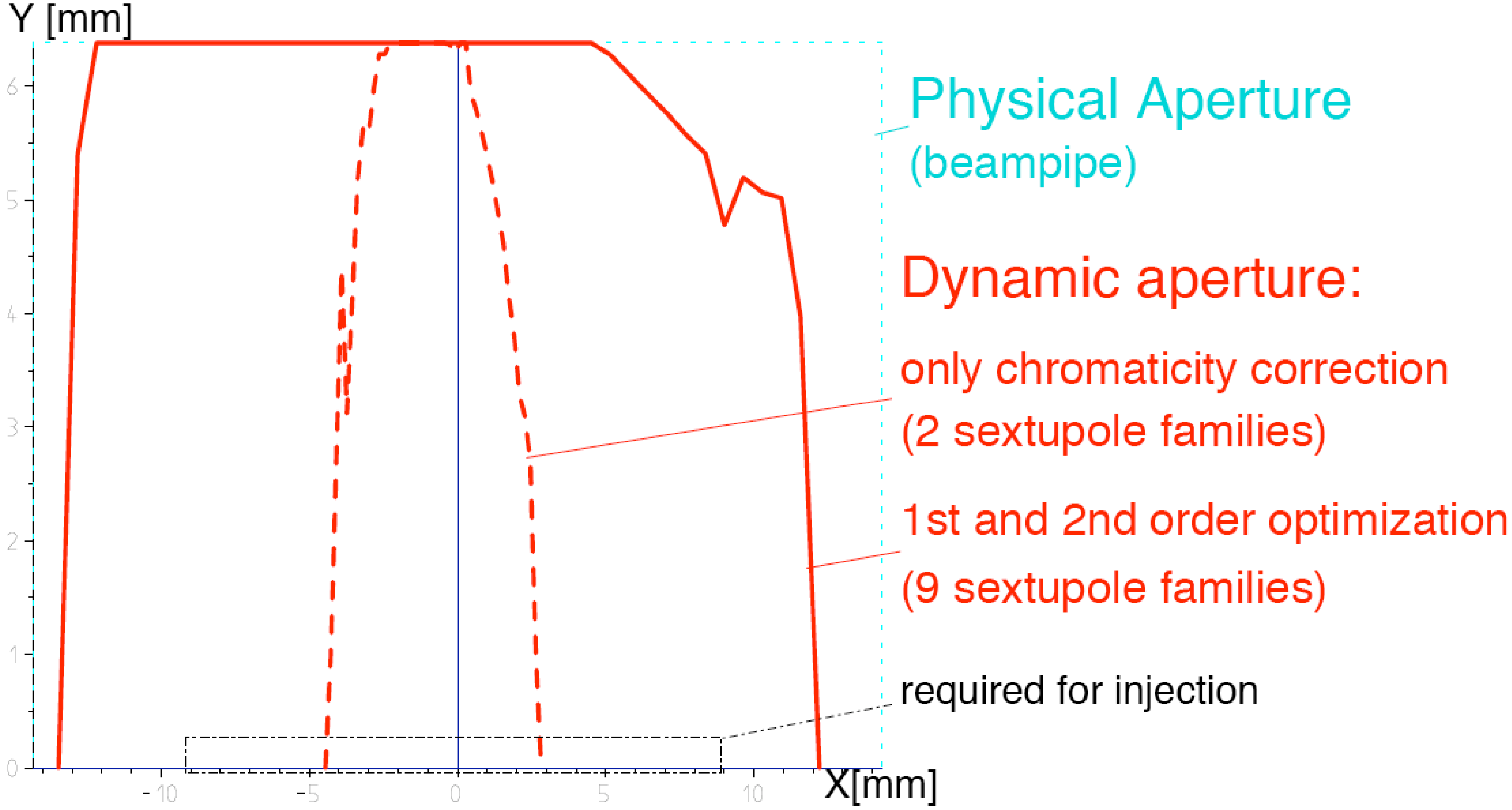
- 2 chromaticities ξ_x, ξ_y
- 2 off-momentum resonances $2Q_x, 2Q_y \rightarrow d\beta/d\delta \rightarrow \xi^{(2)} = \partial^2 Q/\partial\delta^2$
- 2 terms $\longrightarrow$ integer resonances Q_x
- 1 term $\longrightarrow 3^{rd}$ integer resonances $3Q_x$
- 2 terms $\longrightarrow$ coupling resonances $Q_x \pm 2Q_y$

13 second order terms:

- 3 tune shifts with amplitude: $\partial Q_x/\partial J_x, \;\; \partial Q_x/\partial J_y = \partial Q_y/\partial J_x, \;\; \partial Q_y/\partial J_y$
- 8 terms $\longrightarrow$ octupole like resonances: $4Q_x, 2Q_x \pm 2Q_y, 4Q_y, 2Q_x, 2Q_y$
- 2 second order chromaticities: $\partial^2 Q_x/\partial\delta^2$ and $\partial^2 Q_y/\partial\delta^2$

Optimization of Dynamic aperture

- Keep chromaticity sextupole strength low
- Include sextupoles in quadrupoles for more flexibility
- Try an interleaved sextupole scheme (-I transformer)
- Choose working point far from systematic resonances
- Iterate between linear and non-linear lattice



Frequency Map analysis

- Oscillating electrons in storage ring generally obey "quasi-harmonic" motion close to the origin for a "good working point"
- Large amplitudes sample more non-linear fields and motion becomes chaotic - i.e., the frequency of oscillation (tune) changes with turn number.
- Motion close to a resonance also exhibits diffusion
- Frequency map analysis examines dynamics in frequency space rather than configuration space.
- Regular or quasi-regular periodic motion is associated to unique tune values in frequency space
- Irregular motion exhibits diffusion in frequency space, i.e. tunes change
- The mapping of configuration space (x and y) to frequency space (Q_x and Q_y) is regular for regular motion and irregular for chaotic motion.
- Numerically integrate the equations of motion for a set of initial conditions (x, y, x', y') and compute the frequencies as a function of time
- Small amplitude, regular motion
- Large amplitude, chaotic motion and particle loss

NAFF algorithm

Quasi-periodic approximation through **NAFF** algorithm

$$f'_j(t) = \sum_{k=1}^{N} a_{j,k} e^{i\omega_{j,k} t}$$

of a complex phase space function $f_j(t) = q_j(t) + ip_j(t)$

defined over $t = \tau$,

for each degree of freedom $j = 1, \ldots, n$ with $\omega_{j,k} = \boldsymbol{k}_j \cdot \boldsymbol{\omega}$

and $a_{j,k} = A_{j,k} e^{i\phi_{j,k}}$

<u>Advantages of NAFF</u>:

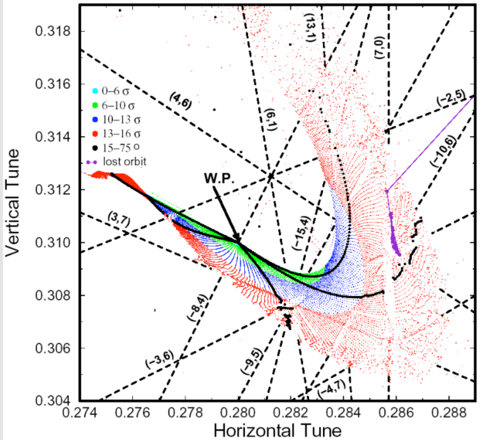
a) Very accurate representation of the "signal" $f_j(t)$ (if quasi-periodic) and thus of the amplitudes $a_{j,k}$

b) Determination of frequency vector $\boldsymbol{\omega} = 2\pi\boldsymbol{\nu} = 2\pi(\nu_1, \nu_2, \ldots, \nu_n)$ with high precision $\frac{1}{\tau^4}$ for Hanning Filter

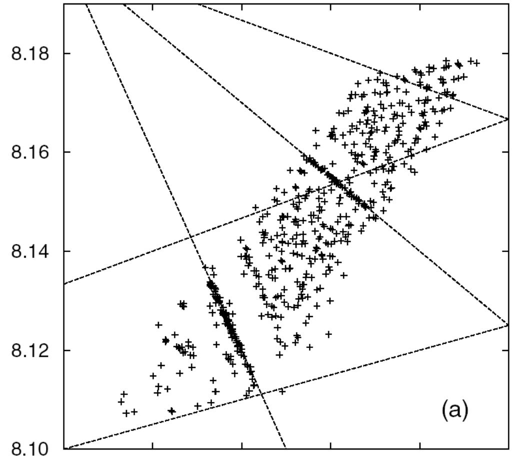
Aspects of frequency map analysis

- Construction of frequency map

$$\mathcal{F}_\tau : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{q}|_{\boldsymbol{p}=\boldsymbol{p}_0} & \longrightarrow & \boldsymbol{\nu} \end{array}$$



LHC Simulations
Papaphilippou PAC99



ALS Measurements
Robin et al. PRL2000

- Determination of tune diffusion vector

$$\boldsymbol{D}|_{t=\tau} = \boldsymbol{\nu}|_{t\in(0,\tau/2]} - \boldsymbol{\nu}|_{t\in(\tau/2,\tau]}$$

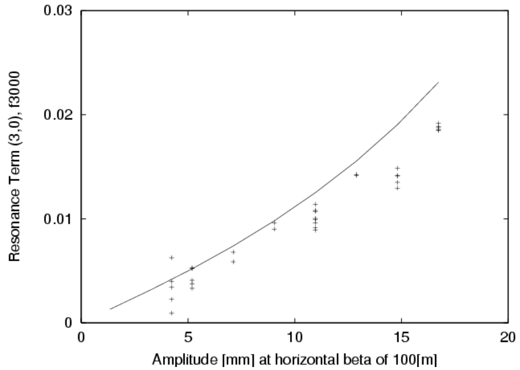
and construction of diffusion map

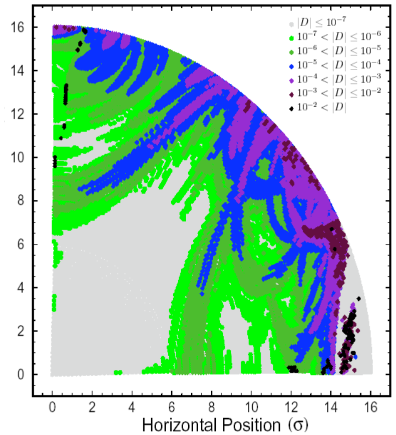
$$\mathcal{D}_\tau : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{q}|_{\boldsymbol{p}=\boldsymbol{p}_0} & \longrightarrow & \boldsymbol{D} \end{array}$$

- Determination of resonance driving terms associated with amplitudes $a_{j,k}$

Bengtsson PhD thesis CERN88-05

SPS Measurements
Bartolini et al. PAC99

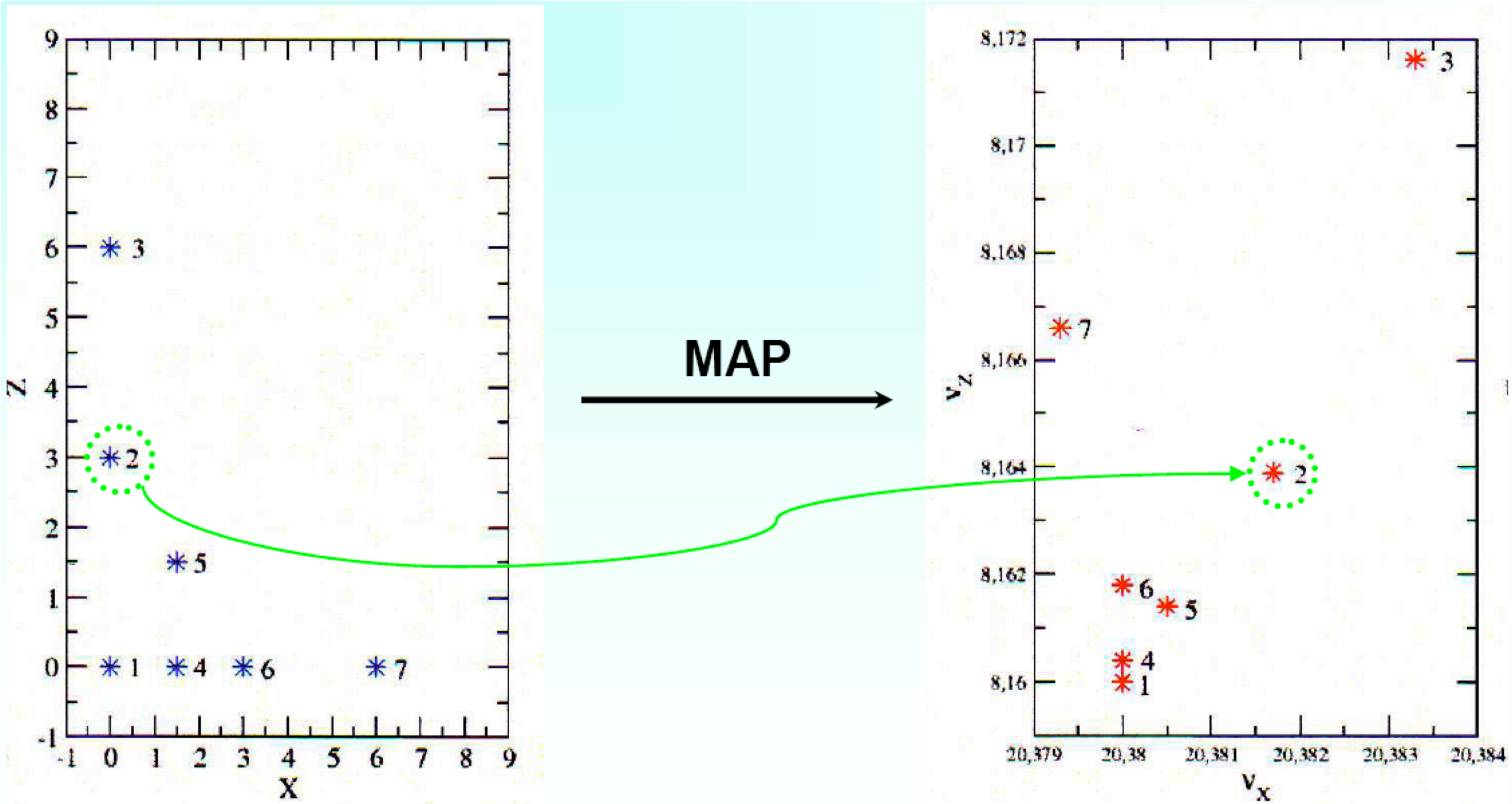




LHC Simulations
Papaphilippou PAC99

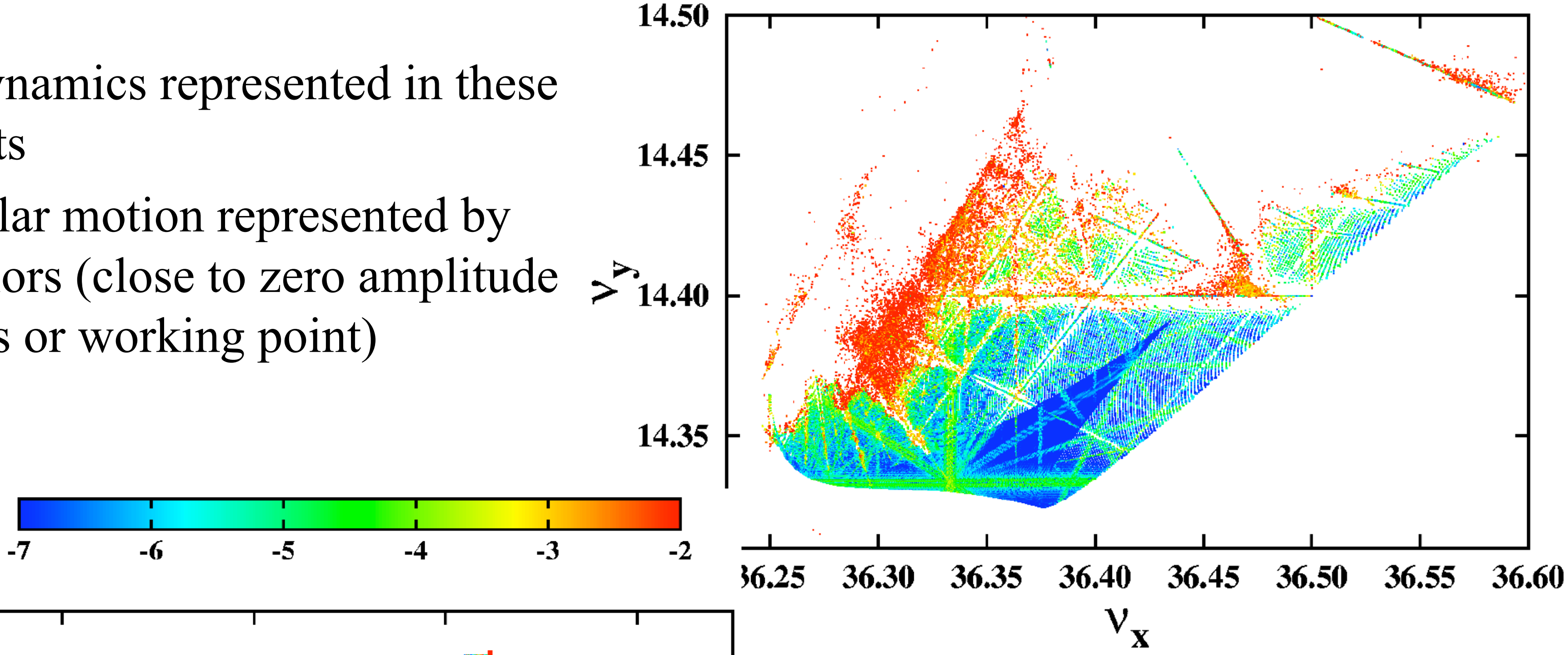
Building the frequency map

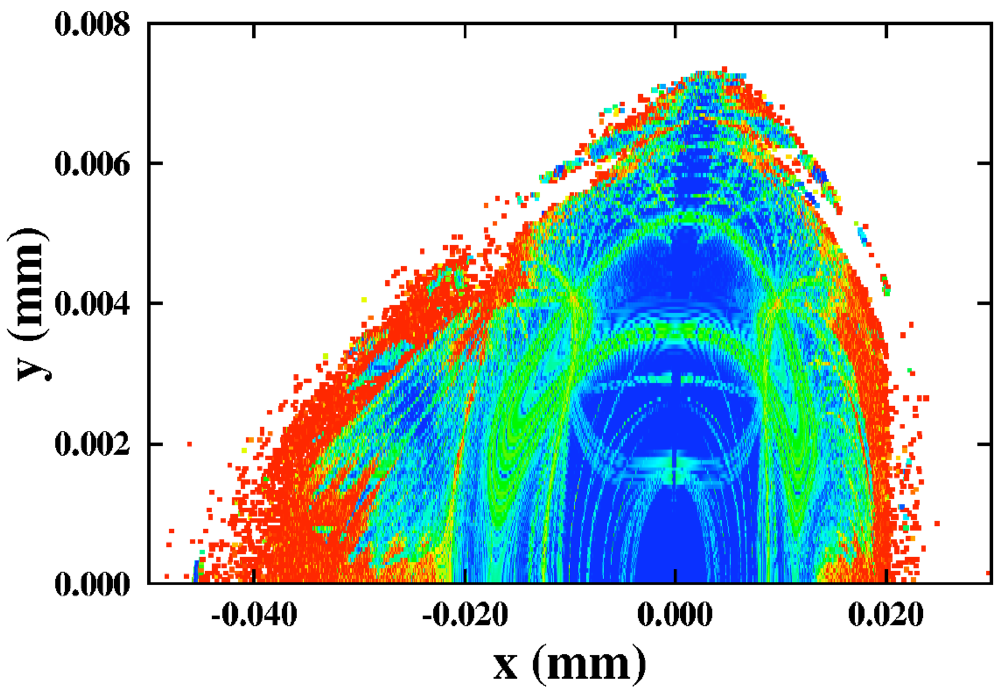
- Choose coordinates (x_i, y_i) with p_x and $p_y=0$
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram



Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)





- Resonances appear as distorted lines in frequency space (or curves in initial condition space
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine

References

- O. Bruning, Non-linear dynamics, JUAS courses, 2006.