BROOKHAVEN NATIONAL LABORATORY

Introduction to Accelerator Dynamics

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In memory of Chronis Polymilis (1946-2000)

“Non-linear Dynamics: Chaos and Complexity”

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Outline

- Overview of accelerator complexes
- Overview of electro-magnetic accelerator components
- The single-particle relativistic Hamiltonian
- Linear betatron motion and action-angle variables
- Generalized non-linear Hamiltonian
- Classical perturbation theory
  - Canonical perturbation method
  - Application to the accelerator Hamiltonian
  - Resonance driving terms and tune-shift

Outline (cont.)

- The single resonance treatment
  - “Secular” perturbation theory and resonance overlap criterion
  - Application to the accelerator Hamiltonian - Resonance widths
- The choice of the working point
- Dynamic aperture
- High-order perturbation theory
  - One-Turn accelerator maps (Lie formalism)
  - Normal form construction
  - Application to the LHC at injection

Outline (cont.)

- Frequency Analysis
  - Description of the method
  - Construction of frequency and diffusion maps
  - Example 1: Application to the LHC (correction, beam/beam effect)
  - Example 2: Application to the SNS (working point choices)
  - Experimental frequency maps
  - Experimental resonance driving terms estimation

Accelerator complexes

- High-energy accelerators[a]
  - Hadron colliders (TeVatron, RHIC, HERA, LHC,VLHC)
  - Lepton colliders (LEP, SLC, PEPII, KEKB, DAFNE, TESLA, NLC, TESLA)
  - Muon collider
- High-intensity beam accumulators
  - Spallation neutron sources (ISIS, SNS, ESS, JAERI)
  - Neutrino factories
  - Nuclear waste transmutation facilities, energy amplifiers
- Sychnotron light sources (ESRF, ALS, PSI, NSLS, SPEAR, SOLEIL, DIAMOND, FELs)

[a] Approved, Operating, Proposed, Dismantled

Relativistic Heavy Ion Collider complex at BNL

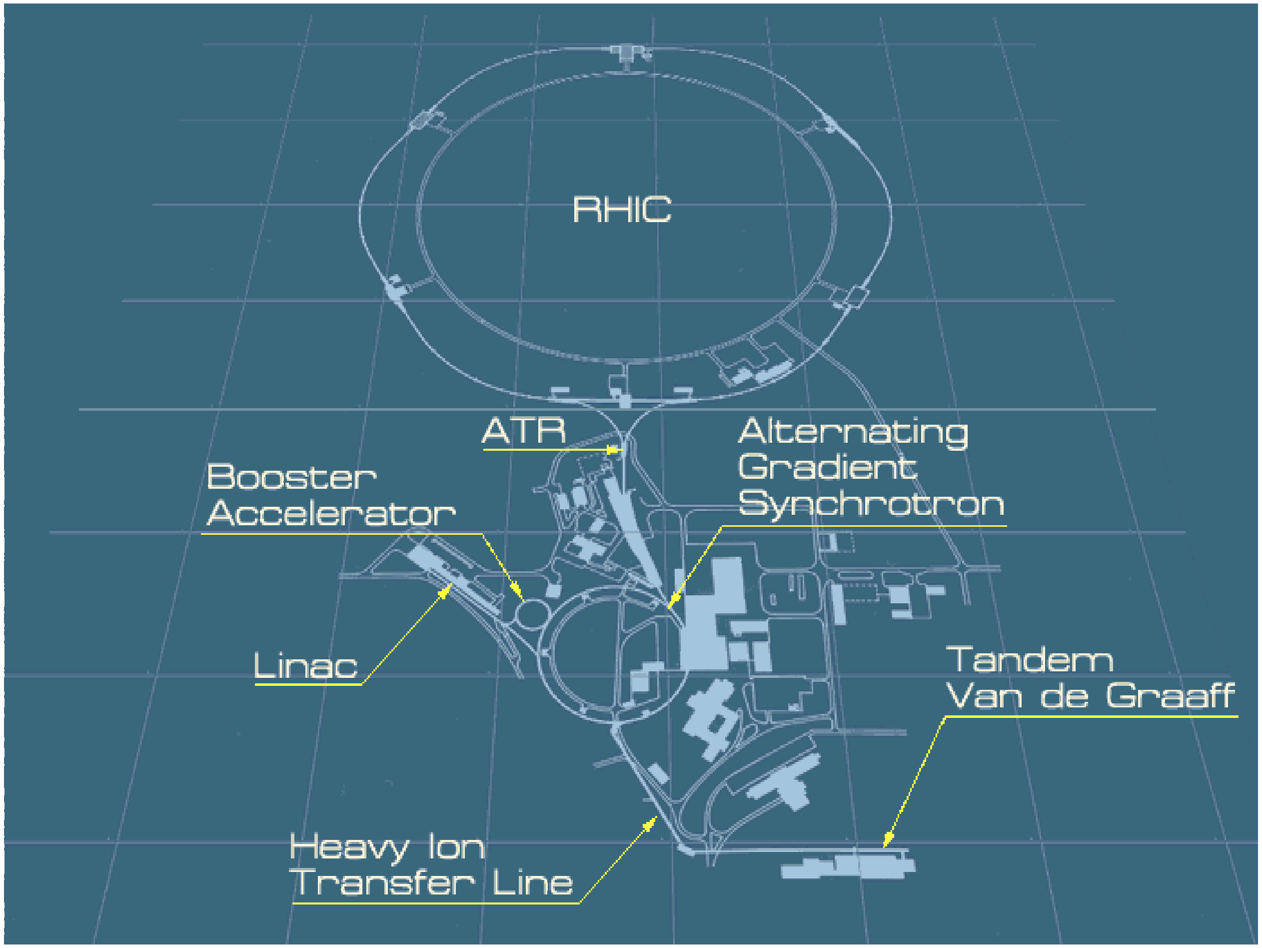

Figure 1: Schematic view of the RHIC complex.

RHIC (cont.)

Figure 2: Photo of the RHIC tunnel with the dipole cryomodules.

TeVatron at Fermilab

Figure 3: Aerial view of the TeVatron.

Large Hadron Collider at CERN

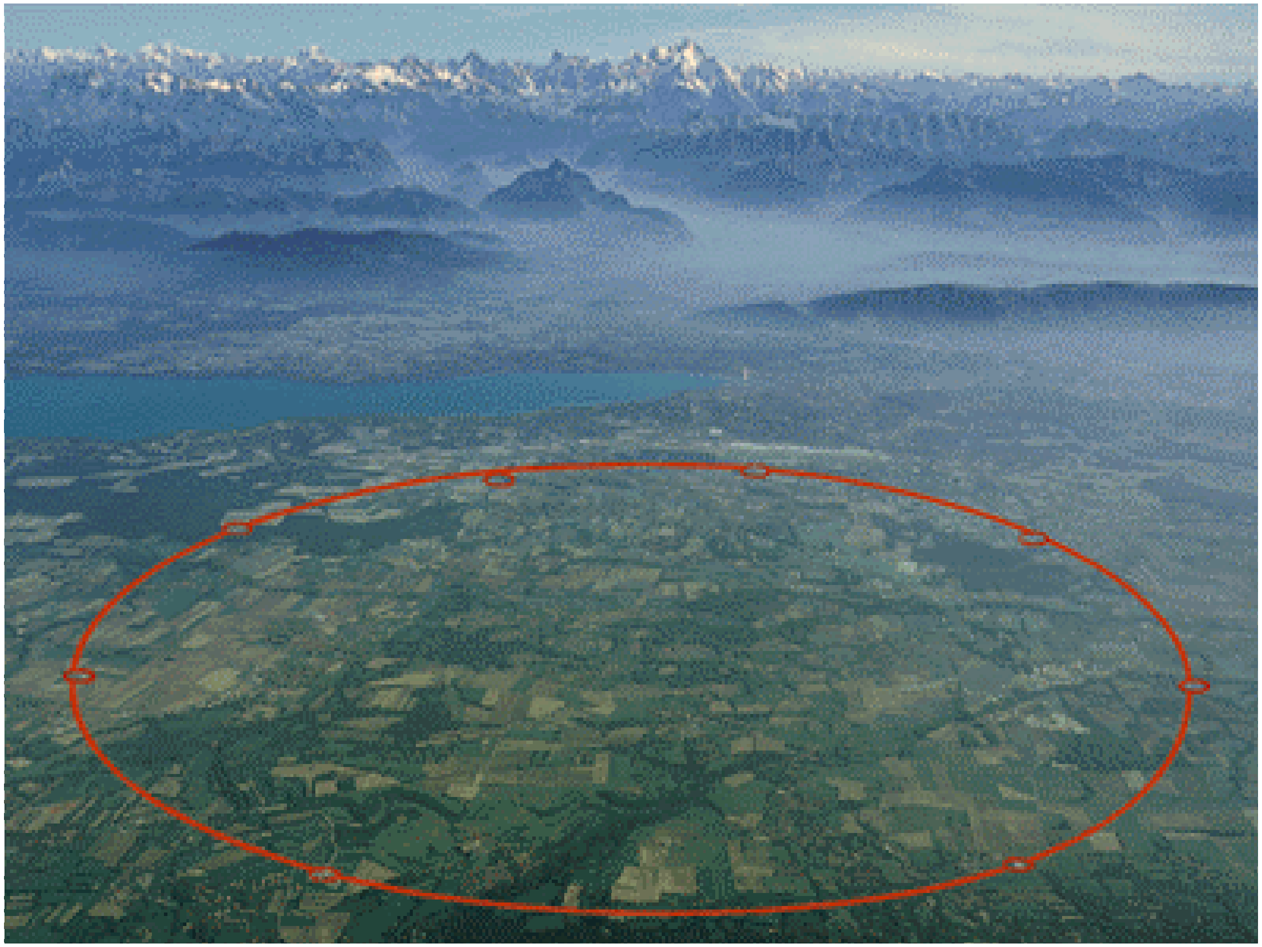

Figure 4: Aerial view of the LHC ring currently under construction in the LEP tunnel.

ISIS Complex

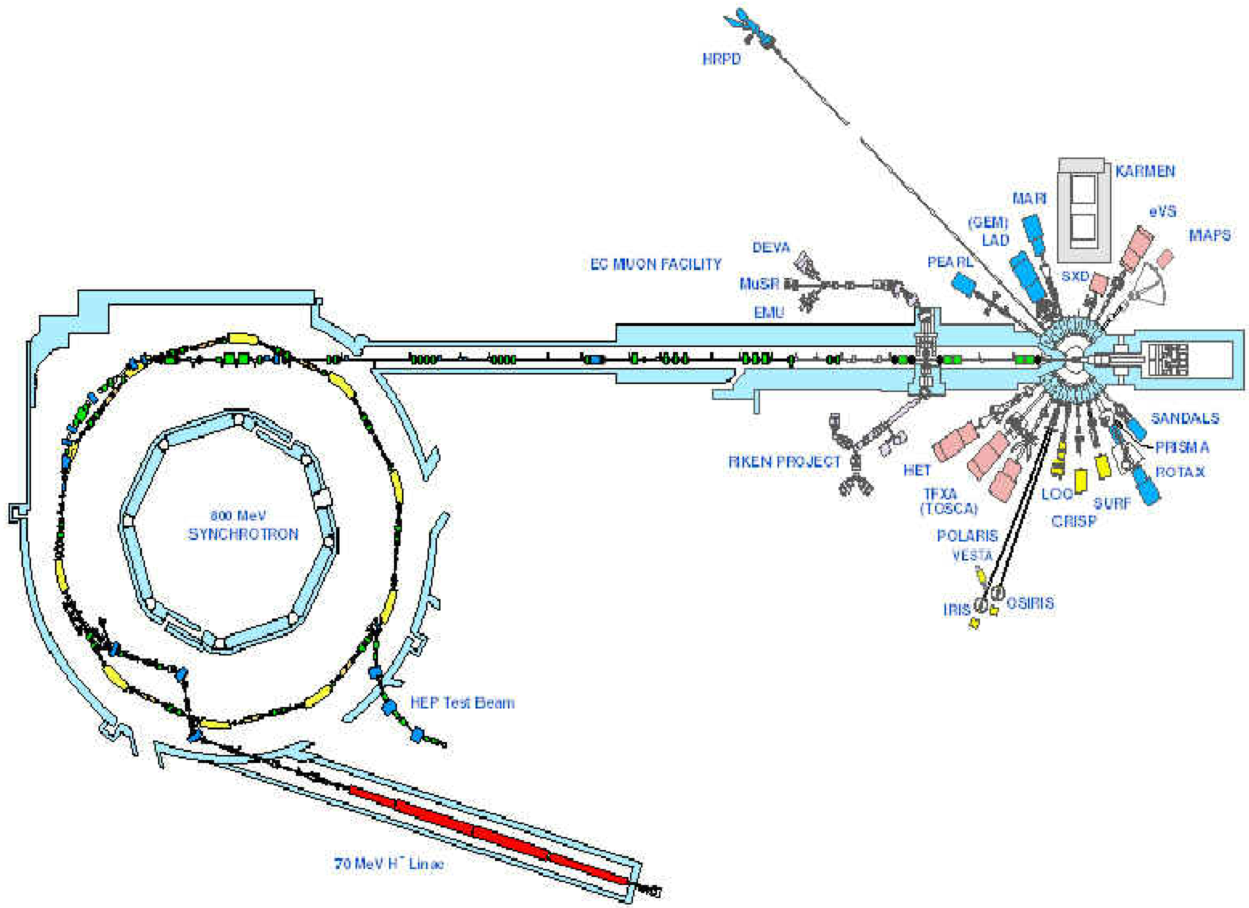

Figure 5: ISIS accelerator complex schematic layout.

Spallation Neutron Source Complex

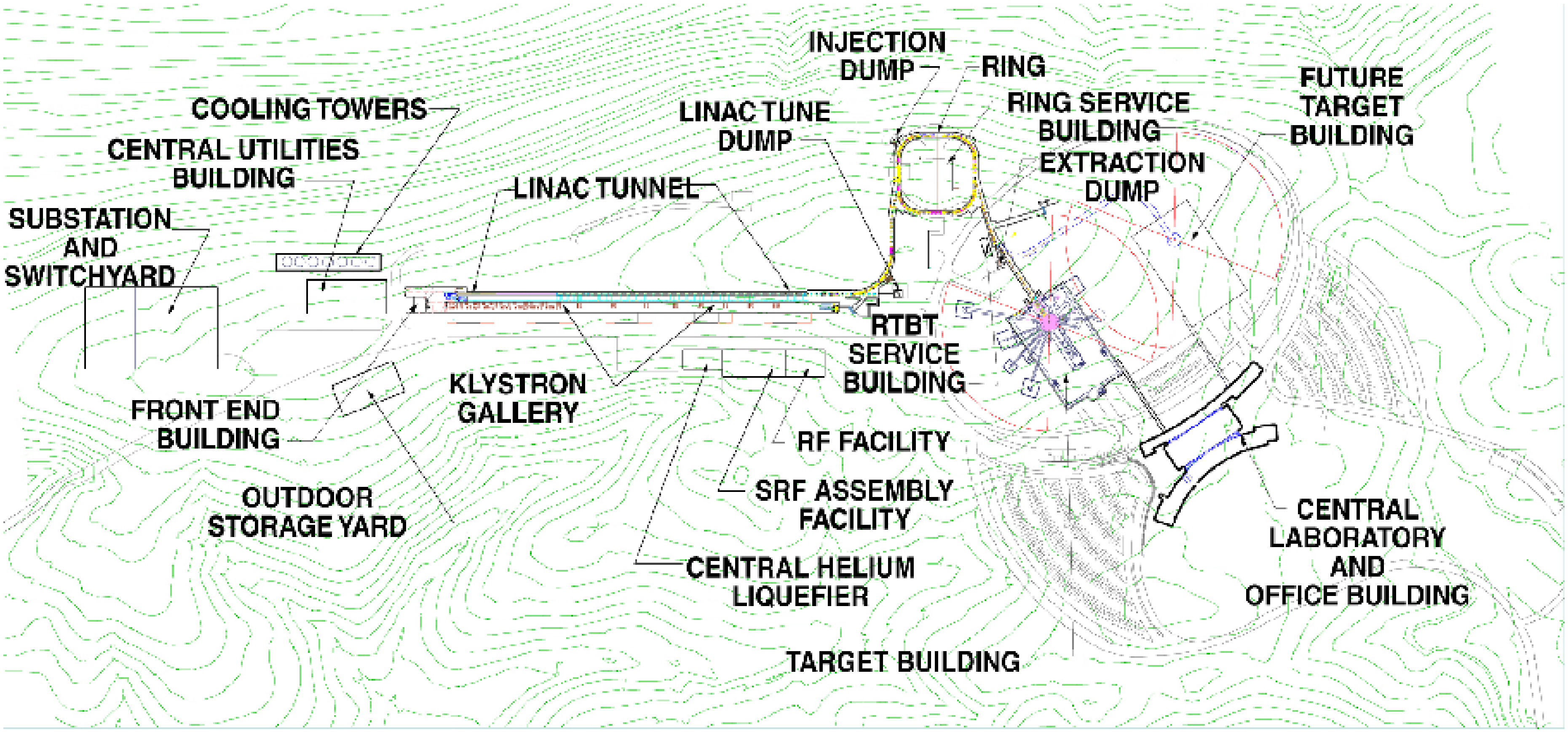

Figure 6: SNS accelerator complex schematic layout.

Accelerator electro-magnetic components

- Main elements
  - Dipoles $\longrightarrow$ Bending of the beam
  - Quadrupoles $\longrightarrow$ (De)Focusing of the beam
  - Sextupoles $\longrightarrow$ Chromaticity correction
  - Octupoles $\longrightarrow$ Linear tune-shift with amplitude
  - RF cavities $\longrightarrow$ Acceleration, longitudinal phase-space manipulation
- Other elements
  - Kickers $\longrightarrow$ Injection, extraction
  - Solenoids $\longrightarrow$ Helical trajectories, electron cloud removal
  - Wigglers $\longrightarrow$ Sychnotron radiation, emittance manipulation
  - Skew quadrupoles $\longrightarrow$ Linear coupling of transverse motion
  - Correction (normal/skew) N-poles $\longrightarrow$ Correction schemes

SNS half-cell assembly

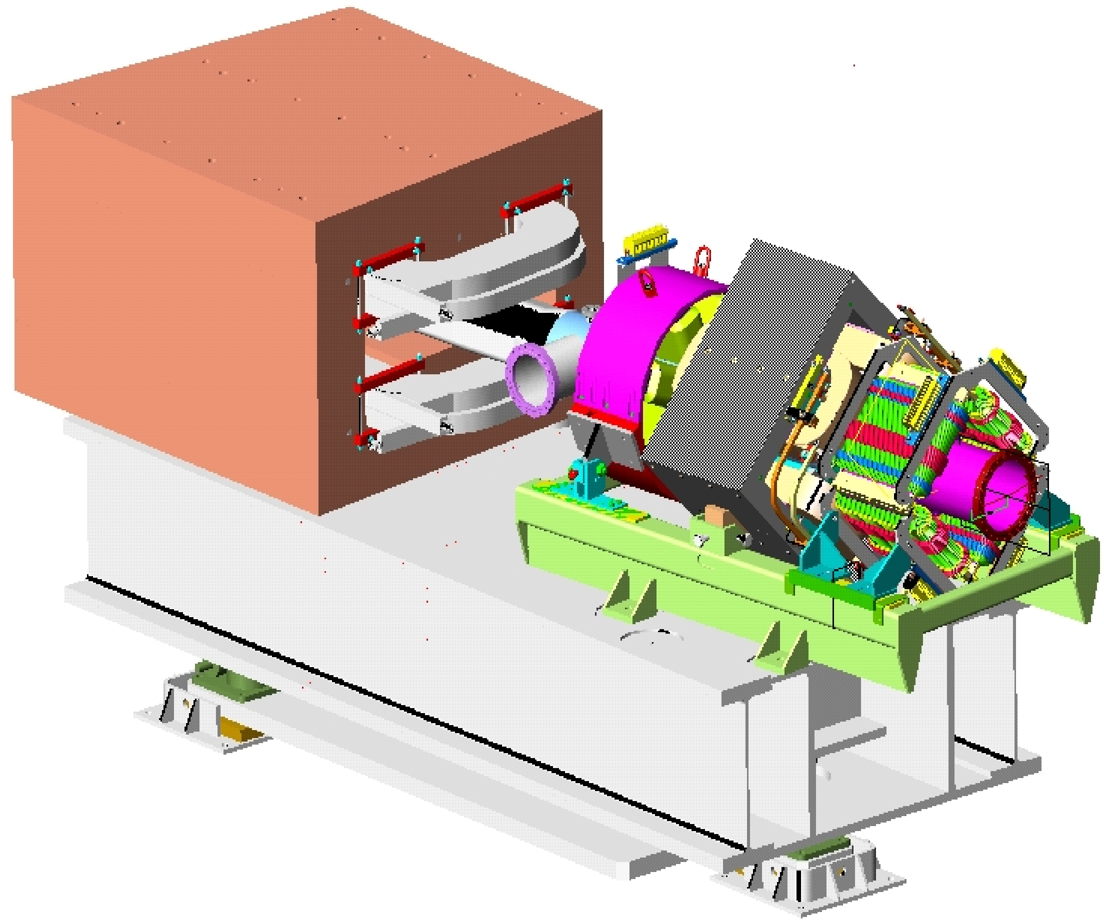

Figure 7: SNS half-cell assembly schematic layout, with the main dipole, chromaticity sextupole, quadrupole and dipole/skew-quadrupole/skew-sextupole corrector.

SNS dipole

Figure 8: SNS 17D120 dipole magnet prototype

SNS corrector

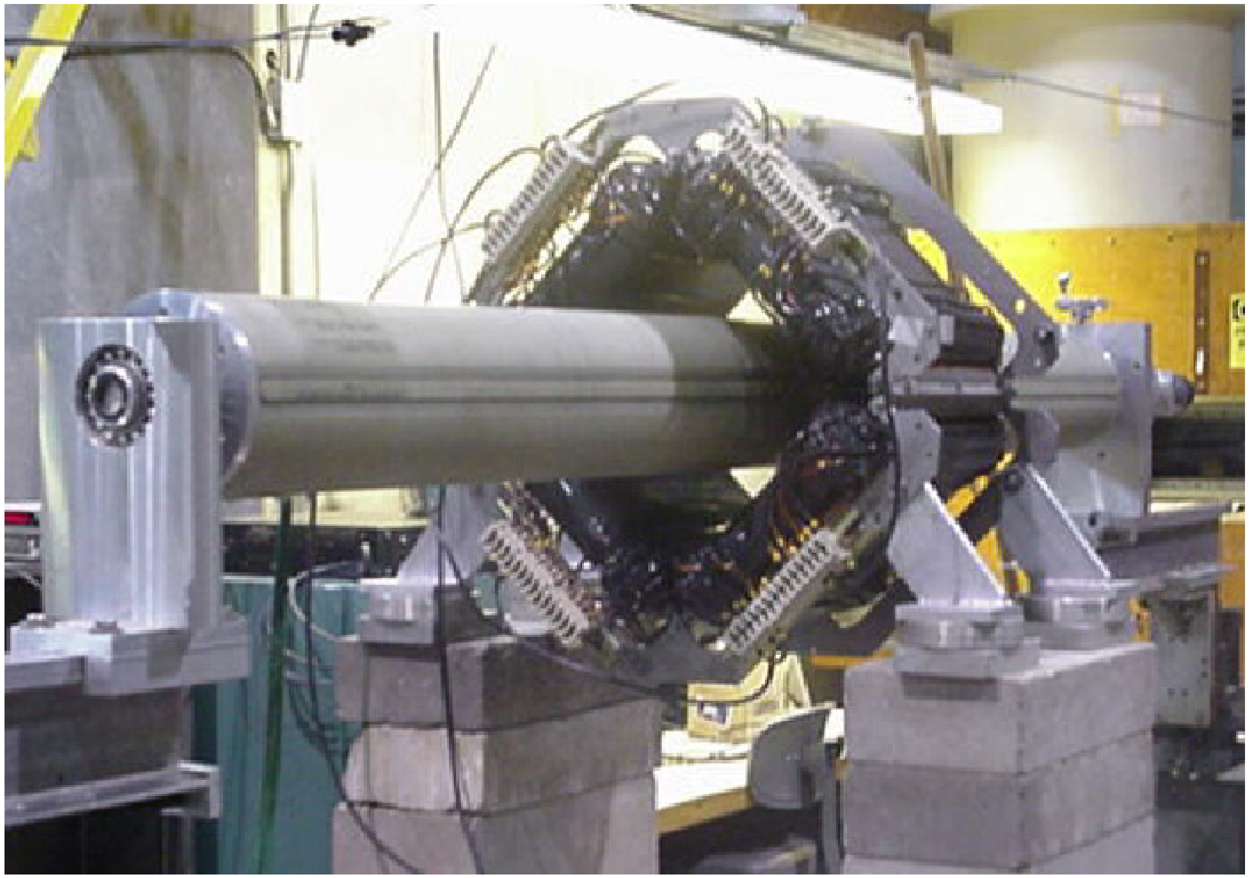

Figure 9: SNS corrector magnet prototype, with dipole, skew quadrupole and skew sextupole components.

LHC Dipole

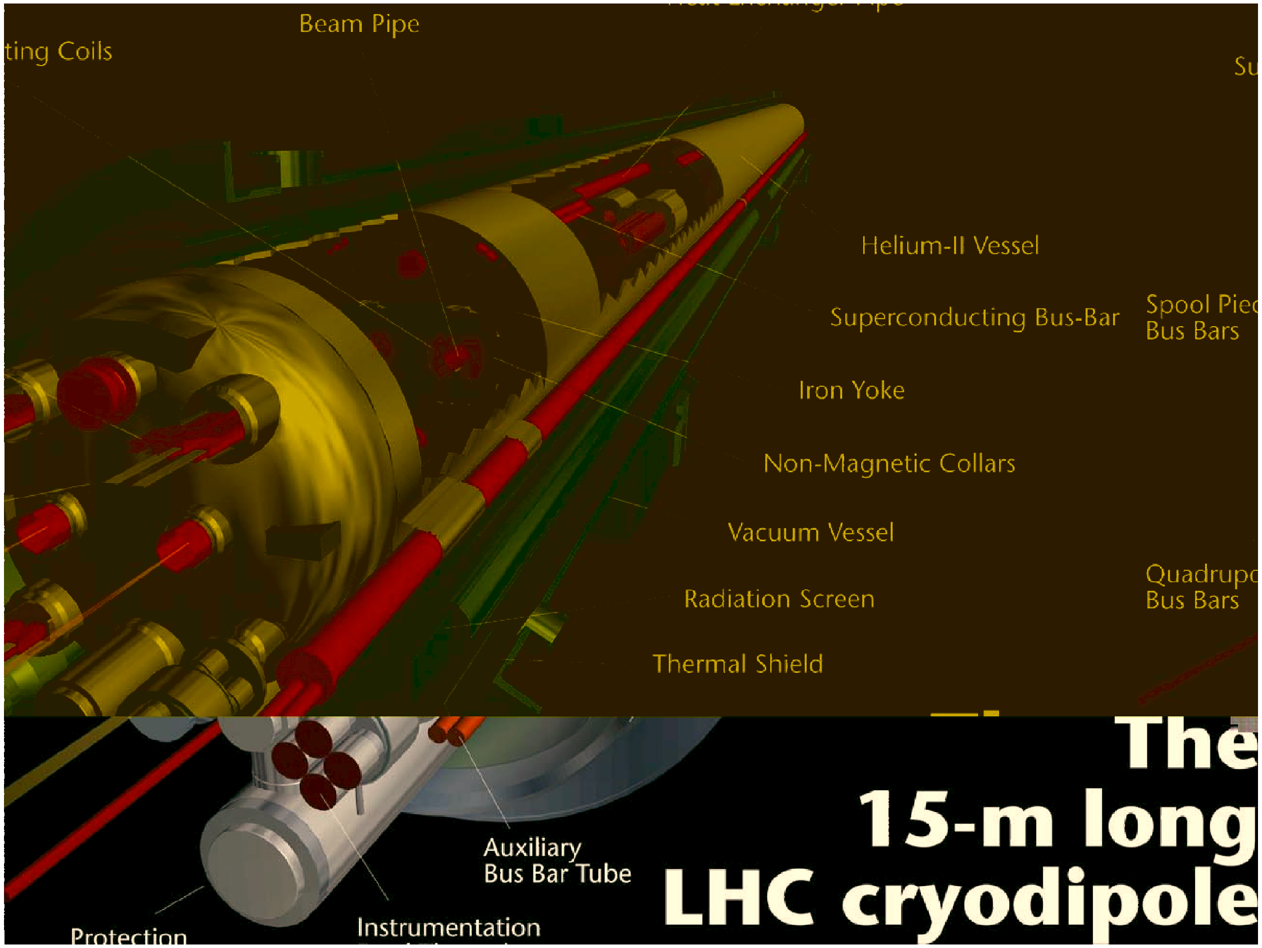

Figure 10: Schematic layout of the LHC main dipole cryomodule

LHC Dipole (cont.)

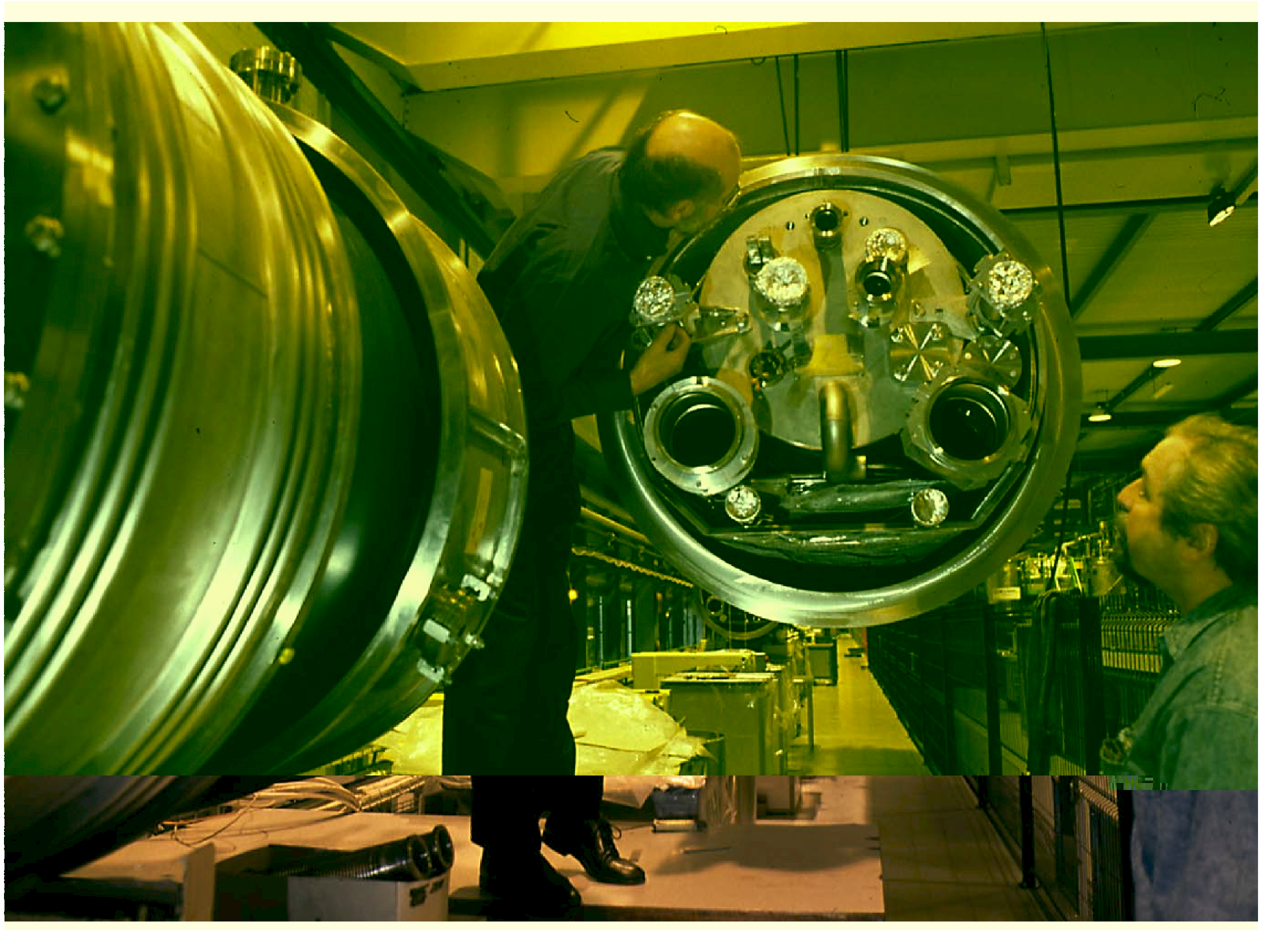

Figure 11: Photo of the LHC main dipole prototype

Non-linear dynamics studies in accelerators

Accelerator design focuses on high **performance**

- Colliders $\longrightarrow$ Luminosity (number of events/second)
  $L = N^2 k_b \gamma/(4\pi\epsilon_n\beta^*)$, with N the number of particles, k_b the number of bunches, $\gamma = E/(m_0c^2)$ the relativistic factor, ϵ_n the normalized emittance (area of the beam), β^* betatron function at the interaction point (oscillation amplitude of the beam)
- High-intensity machines $\longrightarrow$ Average beam power $\bar{P} = \bar{I}E = f_N NeE$, with $\bar{I}$ the average current

Non-linear effects limit the performance $\longrightarrow$ beam loss

- Colliders $\longrightarrow$ Reduced lifetime
- High-intensity machines $\longrightarrow$ Radio-activation

$\downarrow$

Identification of non-linearities and correction

Non-linear effects in accelerators

- Single-Particle effects
  - Errors in magnet design (geometric, systematic, random)
  - Magnet fringe-fields
  - Magnet misalignments and rolls
  - Ground motion
  - Power supplies' ripple
  - Beam-beam effect (weak-strong approximation)
  - Incoherent space-charge force (particle core model)
- Multi-particle effects
  - Beam-beam effect (strong-strong approximation)
  - Coherent space-charge effect
  - Wall effects - wake fields (impedances)
  - Particle scattering (Touschek effect,intra-beam scattering)
  - Other collective instabilities (head-tail, electron cloud, etc.)

The single-particle relativistic Hamiltonian

$$H(\mathbf{x}, \mathbf{p}, t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x}, t)$$

- $\mathbf{x} = (x, y, z)$ Cartesian positions
- $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
- $\mathbf{A} = (A_x, A_y, A_z)$ the magnetic vector potential
- Φ the electric scalar potential
- c and e velocity of light and particle charge

The ordinary kinetic momentum vector

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

with $\mathbf{v}$ the particle velocity and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor.

The single-particle relativistic Hamiltonian (cont.)

The Hamiltonian is the total energy

$$H \equiv E = \gamma m c^2 + e\Phi$$

The total kinetic momentum

$$P = \left(\frac{H^2}{c^2} - m^2 c^2 \right)^{1/2}$$

Use Hamilton's equations $(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = [(\mathbf{x}, \mathbf{p}), H]$ to get equations of motion

$$\downarrow$$

Lorenz equations

The single-particle relativistic Hamiltonian (cont.)

Canonical Transformation **I**: $(x, y, z, p_x, p_y, p_z) \mapsto (x, y, s, p_x, p_y, p_s)$

moving the coordinate system on a closed curve, with path length s, of a particle with reference momentum P_0 in the guiding magnetic field.

The new Hamiltonian

$$H(\mathbf{x}', \mathbf{p}', t) = c\sqrt{(p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_x)^2 + \frac{(p_s - \frac{e}{c}A_s)^2}{(1 + \frac{x}{\rho(s)})^2} + m^2c^2} + e\Phi(\mathbf{x}')$$

- $A_s = (\mathbf{A} \cdot \hat{\mathbf{s}})(1 + \frac{x}{\rho(s)})$
- $p_s = (\mathbf{p} \cdot \hat{\mathbf{s}})(1 + \frac{x}{\rho(s)})$
- $\rho(s)$ the local radius of curvature

The single-particle relativistic Hamiltonian (cont.)

Canonical Transformation **II**: $(x, y, s, p_x, p_y, p_s) \mapsto (x, y, t, p_x, p_y, -H)$

setting the path s as the independent variable and the momentum $-p_s$ as the new Hamiltonian. Assuming $\Phi = 0$

$$\mathcal{H} \equiv (-p_s) = -\frac{e}{c}A_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{\frac{H^2}{c^2} - m^2c^2 - (p_x - \frac{e}{c}A_x)^2 - (p_y - \frac{e}{c}A_y)^2}$$

If $\mathbf{A}$ time independent $\longrightarrow \mathcal{H}$ integral of motion, i.e. constant along the particle trajectories.

$\star\star\star$ Longitudinal motion neglected (not valid when accelerating) $\star\star\star$

Linear betatron motion

Assume a simple case of linear transverse fields:

$$\begin{aligned} B_x &= \quad b_1(s)y \\ B_y &= -\,b_0(s) + b_1(s)x \end{aligned} \quad ,$$

- main bending field $B_0 \equiv b_0(s) = \frac{P_0 c}{e\rho(s)}$ [T]
- normalized quadrupole gradient $K(s) = b_1(s)\frac{e}{cP_0} = \frac{b_1(s)}{B\rho}$ $[1/\mathrm{m}^2]$
- magnetic rigidity $B\rho = \dfrac{P_0 c}{e}$ $[\mathrm{T}\cdot\mathrm{m}]$

The vector potential $\mathbf{A} = (0, 0, A_s)$ with

$$A_s(x,y,s) = -\frac{P_0 c}{e}\left[\frac{x}{\rho(s)} + \left(\frac{1}{\rho(s)^2} - K(s)\right)\frac{x^2}{2} + K(s)\frac{y^2}{2}\right] \quad .$$

Linear betatron motion (cont.)

The Hamiltonian can be written as:

$$\mathcal{H} = -P_0 \left[\frac{x}{\rho(s)} + \left(\frac{1}{\rho^2} - K(s) \right) \frac{x^2}{2} + K(s) \frac{y^2}{2} \right] - \left(1 + \frac{x}{\rho(s)} \right) \sqrt{P^2 - p_x^2 - p_y^2}$$

Equations of motion still non-linear in the canonical momenta!

- Canonical transformation **III**: $(p_x, p_y, -H) \mapsto (\frac{p_x}{P}, \frac{p_y}{P}, \frac{-H}{P})$ rescaling the canonical momenta
- Expand the square root
- Throw away terms higher than quadratic (not always a good idea!)
- Introduce the momentum spread $\delta = \frac{\delta P}{P_0} = (\frac{P - P_0}{P_0})$
- New canonical momentum $p_x = \frac{dx}{ds} \equiv x'$

Linear betatron motion (cont.)

The new Hamiltonian:

$$\hat{\mathcal{H}} \equiv (-p_s/P) = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{\delta}{1+\delta}\frac{x}{\rho(s)} + \frac{1}{1+\delta}\left[\left(\frac{1}{\rho(s)^2} - K(s)\right)\frac{x^2}{2} + K(s)\frac{y^2}{2}\right]$$

Hill's equations of betatron motion

$$x'' + \frac{1}{1+\delta}\left(\frac{1}{\rho(s)^2} - K(s)\right)x = \frac{\delta}{1+\delta}\frac{1}{\rho(s)}$$
$$y'' + \frac{1}{1+\delta}K(s)y = 0$$

Introduce dispersion function

$$D_x'' - \frac{1}{1+\delta}K_x(s)D_x = \frac{1}{\rho(s)} \quad ,$$

where $K_x(s) \equiv \dfrac{1}{\rho(s)^2} - K(s)$ and $K_y(s) \equiv K(s)$.

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Linear betatron motion (cont.)

- Canonical transformation **IV**: $(x, y, p_x, p_y) \mapsto (x - x_0, y, p_x - p_{x0}, p_y)$

shifting the coordinate system's origin to the closed orbit.
The new equations of motion

$$x'' + K_x(s)x, = 0 \quad , \quad y'' + K_y(s)y = 0 \ .$$

Homogeneous equations with periodic coefficients $K_{x,y}(s) = K_{x,y}(s + C)$

Floquet Theory

$$\downarrow$$

$$x = \sqrt{A_x\beta_x(s)}\cos(\psi_x(s) + \psi_{0x}) \quad , \quad y = \sqrt{A_y\beta_y(s)}\cos(\psi_y(s) + \psi_{0y}) \ .$$

- the Courant-Snyder invariant $A_{x,y}$
- the Courant-Snyder amplitude function $\beta_{x,y}(s)$
- the alpha $\alpha_{x,y} = -\frac{\beta'_{x,y}}{2}$ and gamma $\gamma_{x,y} = \frac{1+\alpha^2_{x,y}}{\beta'_{x,y}}$ functions
- the phase advance $\psi_{x,y}(s) = \int_0^s \frac{d\tau}{\beta_{x,y}(\tau)}$
- the tunes $Q_{x,y} = \frac{1}{2\pi}\int_0^C \frac{ds}{\beta_{x,y}(s)}$

Action-angle variables

- Canonical transformation **V** to action angle variables $(\phi_x, \phi_y, J_x, J_y)$

$$x(s) = \sqrt{2\beta_x(s)J_x}\cos(\phi_x(s) + \theta_x(s))$$

$$p_x(s) = -\sqrt{\frac{2J_x}{\beta_x(s)}}[\sin(\phi_x(s) + \theta_x(s)) + \alpha_x(s)\cos(\phi_x(s) + \theta_x(s))]$$

with

$$\theta_x(s) = -\arctan\left[\frac{\beta_x(s)x' + \alpha_x(s)x}{x}\right] - \phi_x(s) \quad .$$

The new Hamiltonian

$$H_0(J_x, J_y) = \frac{1}{R}(Q_x J_x + Q_y J_y) \quad ,$$

The equations of motion

$$J_x = \text{constant} \quad , \qquad J_y = \text{constant}$$

$$\phi_x(s) = \phi_{x0}(s) + \frac{Q_x(s - s_0)}{R} \quad , \qquad \phi_y(s) = \phi_{y0}(s) + \frac{Q_y(s - s_0)}{R} \quad ,$$

and describe a 2-torus in the phase space $(\phi_x, \phi_y, J_x, J_y)$.

Generalized “non-integrable” Hamiltonian

Transverse vector potential:

- Basic law of magnetostatics $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \longrightarrow \exists\, \boldsymbol{A}\ : \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
- Ampére’s law in vacuum $\boldsymbol{\nabla} \times \boldsymbol{B} = 0 \longrightarrow \exists\, V\ : \boldsymbol{B} = \boldsymbol{\nabla} V$

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_z}{\partial y} \quad , \quad B_y = -\frac{\partial V}{\partial y} = \frac{\partial A_z}{\partial s}$$

$$\uparrow$$

Cauchy-Riemann conditions of analytic functions

$$\downarrow$$

$$\mathcal{A}(x+iy) = A_z(x,y) + iV(x,y) = \sum_{n=1}^{\infty} (\kappa_n + i\lambda_n)(x+iy)^n$$

The normal and skew multipole coefficients

$$b_{n-1} = -\frac{n\kappa_n r_0^{n-1}}{B_0} \qquad \text{and} \qquad a_{n-1} = \frac{n\lambda_n r_0^{n-1}}{B_0}$$

with r_0 the reference radius, B_0 the main field

Generalized “non-integrable” Hamiltonian (cont.)

- The Hamiltonian:

$$\mathcal{H} \equiv (-p_s/P) = \frac{1}{2}\left(p_x^2 + p_y^2\right) - \frac{x}{\rho(s)} - \frac{e}{cP}A_s$$

The vector potential component A_s is:

$$A_s = (\mathbf{A}\cdot\hat{\mathbf{s}})(1 + \frac{x}{\rho(s)})$$

$$= (1 + \frac{x}{\rho(s)})B_0\Re e\sum_{n=0}^{\infty}\frac{b'_n - ia'_n}{n+1}(x+iy)^{n+1}$$

Use transformation **IV** $x = x_\beta + D_x\delta + x_0,\ y = y_\beta + D_y\delta + y_0$

The new Hamiltonian is

$$\mathcal{H}' = H_0 + \sum_{k_x,k_y} h_{k_x,k_y}(s)x^{k_x}y^{k_y}$$

where H_0 is the integrable Hamiltonian. The terms $h_{k_x,k_y}(s)$ are periodic and depend on a'_n, b'_n and the closed orbit displacements $\Delta_x = D_x\delta + x_0,\ \Delta_y = D_y\delta + y_0$

Classical perturbation theory

(Linstead (1882), Poincaré (1892), Von-Zeipel (1916))

Consider a general Hamiltonian with n degrees of freedom

$$H(\boldsymbol{J}, \boldsymbol{\varphi}, \theta) = H_0(\mathbf{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\varphi}, \theta) + \mathcal{O}(\epsilon^2)$$

where the non-integrable part $H_1(\boldsymbol{J}, \boldsymbol{\varphi}, \theta)$ is 2π-periodic on θ and φ.

If ϵ small $\longrightarrow$ distorted tori still exist $\longrightarrow$ try to "straighten up" the tori

$$\downarrow$$

Canonical Transformation **VI**: Search a generating function

$$S(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta) = \bar{\boldsymbol{J}} \cdot \boldsymbol{\varphi} + \epsilon S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta) + \mathcal{O}(\epsilon^2)$$

for transforming old variables to $(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})$ so that new $\bar{H}(\bar{\boldsymbol{J}})$ only a function of the new actions.

Classical perturbation theory (cont.)

By the canonical transformation equations, the old action and new angle are:

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \boldsymbol{\varphi}} + \mathcal{O}(\epsilon^2)$$

$$\bar{\boldsymbol{\varphi}} = \boldsymbol{\varphi} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \boldsymbol{\varphi}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

Inverting the previous equations:

$$\boldsymbol{J} = \bar{\boldsymbol{J}} + \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} - \epsilon \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2)$$

$\star\star\star$ S_1 is expressed in terms of the new variables $\star\star\star$

Classical perturbation theory (cont.)

The new Hamiltonian is:

$$\bar{H}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta) = H(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta) + \epsilon\frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\theta} + \mathcal{O}(\epsilon^2)$$

Expand term by term the old Hamiltonian to leading order in ϵ:

$$H_0(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})) = H_0(\bar{\boldsymbol{J}}) + \epsilon\frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial\bar{\boldsymbol{J}}}\frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2)$$

$$\epsilon H_1(\boldsymbol{J}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\boldsymbol{\varphi}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}),\theta) = \epsilon H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}}) + \mathcal{O}(\epsilon^2)$$

Equating the terms of equal order in ϵ, we get in zero order $\bar{H}_0 = H_0(\bar{\boldsymbol{J}})$ and in first order:

$$\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}})\cdot\frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\bar{\boldsymbol{\varphi}}} + H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})$$

with $\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \dfrac{\partial H_0(\bar{\boldsymbol{J}})}{\partial\bar{\boldsymbol{J}}}$ the unperturbed frequency vector.

Classical perturbation theory (cont.)

New Hamiltonian should be a function of $\bar{\boldsymbol{J}}$ only $\longrightarrow$ eliminate $\bar{\boldsymbol{\varphi}}$

$$\downarrow$$

- Average part: $\langle H_1 \rangle_{\bar{\boldsymbol{\varphi}}} = \left(\frac{1}{2\pi}\right)^n \oint H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) d\bar{\boldsymbol{\varphi}}$
- Oscillating part: $\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\boldsymbol{\varphi}}}$

Using the previous equations, $\bar{H}_1$ becomes

$$\bar{H}_1 = \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta)}{\partial \bar{\boldsymbol{\varphi}}} + \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) \rangle_{\bar{\boldsymbol{\varphi}}} + \{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\} \quad .$$

Classical perturbation theory (cont.)

Choose S_1 so that $\bar{\boldsymbol{\varphi}}$ dependence is eliminated:

$$\bar{H}_1(\bar{\boldsymbol{J}}) = \langle H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\rangle_{\bar{\boldsymbol{\varphi}}} \quad \text{and} \quad \frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\theta} + \boldsymbol{\omega}(\bar{\boldsymbol{J}})\cdot\frac{\partial S_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}},\theta)}{\partial\bar{\boldsymbol{\varphi}}} = -\{H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\} \ ,$$

New Hamiltonian is a function of the new actions only to leading order!!!

$$\bar{H}(\bar{\boldsymbol{J}}) = H_0(\bar{\boldsymbol{J}}) + \epsilon\,\langle H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\rangle_{\bar{\boldsymbol{\varphi}}} + \mathcal{O}(\epsilon^2) \ ,$$

with the new frequency vector

$$\bar{\boldsymbol{\omega}}(\bar{\boldsymbol{J}}) = \frac{\partial\bar{H}(\bar{\boldsymbol{J}})}{\partial\bar{\boldsymbol{J}}} = \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + \epsilon\frac{\partial\langle H_1(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})\rangle_{\bar{\boldsymbol{\varphi}}}}{\partial\bar{\boldsymbol{J}}} + \mathcal{O}(\epsilon^2) \ .$$

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Classical perturbation theory (cont.)

BUT can we find an appropriate generating function S_1?

$$\downarrow$$

Expand in Fourier series

$$\{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}})\} = \sum_{\boldsymbol{k},p} H_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} + p\theta)} \quad ,$$

with $\boldsymbol{k} \cdot \bar{\boldsymbol{\varphi}} = k_1 \bar{\varphi_1} + \cdots + k_n \bar{\varphi_n}$. and also

$$S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}, \theta) = \sum_{\boldsymbol{k},p} S_{1\mathbf{k}}(\bar{\boldsymbol{J}}) e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} + p\theta)} \quad .$$

The unknown amplitudes $S_{1k}(\bar{\boldsymbol{J}})$ are:

$$S_{1k}(\bar{\boldsymbol{J}}) = i \frac{H_{1k}(\bar{\boldsymbol{J}})}{\boldsymbol{k} \cdot \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + p} \qquad \text{with} \qquad \boldsymbol{k}, p \neq \mathbf{0} \quad ,$$

and finally

$$S(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\varphi}}) = \bar{\boldsymbol{J}} \cdot \bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k} \neq \mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\boldsymbol{k} \cdot \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + p} e^{i(\boldsymbol{k}\cdot\bar{\boldsymbol{\varphi}} + p\theta)} + \mathcal{O}(\epsilon^2) \quad .$$

Classical perturbation theory (cont.)

- Remarks

1) What about higher orders?

In principle, the technique works for arbitrary order, **but** variables disentangling becomes difficult (even for 2nd order!!!)

$$\downarrow$$

Solution $\longrightarrow$ Lie transformations

(Application in beam dynamics by Dragt and Finn (1976))

2) What about small denominators?

Resonances $\boldsymbol{k} \cdot \boldsymbol{\omega}(\bar{\boldsymbol{J}}) + p = 0$

$$\downarrow$$

Solution $\longrightarrow$ Super-convergent perturbation techniques - KAM theory

(Kolmogorov (1957), Arnold (1962) and Moser (1962))

Application to the accelerator Hamiltonian

(Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

- The general accelerator Hamiltonian:

$$\mathcal{H}'(J_x, J_y, \phi_x, \phi_y) = H_0(J_x, J_y) + H_1(J_x, J_y, \phi_x, \phi_y)$$

The transverse variable:

$$x(s) = \sqrt{\frac{J_x \beta_x(s)}{2}} \left(e^{i(\phi_x(s)+\theta_x(s))} + e^{-i(\phi_x(s)+\theta_x(s))} \right)$$

and the equivalent y. The Hamiltonian in action-angle variables:

$$\mathcal{H}'(J_x, J_y, \phi_x, \phi_y) = H_0(J_x, J_y) + H_1(J_x, J_y, \phi_x, \phi_y)$$

- The integrable part $H_0(J_x, J_y) = \frac{1}{R}(Q_x J_x + Q_y J_y)$
- The perturbation

$$H_1(J_x, J_y, \phi_x, \phi_y; s) = \sum_{k_x, k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_{j}^{k_x} \sum_{l}^{k_y} g_{j,k,l,m}(s) e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

- The coefficients

$$g_{j,k,l,m}(s) = \frac{h_{k_x,k_y}(s)}{2^{\frac{j+k+l+m}{2}}} \binom{k_x}{j} \binom{k_y}{l} \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\theta_x(s)+(l-m)\theta_y(s)]}$$

- The indexes j, k, l, m: $k_x = j + k$ and $k_y = l + m$.

Resonance driving terms

Coefficients $h_{k_x,k_y}(s)$ are periodic in $\theta \longrightarrow$ expand in Fourier series

$$H_1(J_x, J_y, \phi_x, \phi_y; s) = \sum_{k_x,k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_{j}^{k_x} \sum_{l}^{k_y} \sum_{p=-\infty}^{\infty} g_{j,k,l,m;p} e^{i[(j-k)\phi_x + (l-m)\phi_y - p\frac{s}{R}]}$$

with the **resonance driving terms**

$$g_{j,k,l,m;p} = \binom{k_x}{j}\binom{k_y}{l} \frac{1}{2^{\frac{j+k+l+m}{2}}} \frac{1}{2\pi} \oint h_{k_x,k_y}(s) \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\theta_x(s) + (l-m)\theta_y(s) + p\frac{s}{R}]}$$

For $n_x = j - k$ and $n_y = l - m$ we have the resonance conditions

$n_x Q_x + n_y Q_y = p$.

Goal of accelerator design and correction systems $\longrightarrow$ minimize $g_{j,k,l,m;p}$ by

- Change magnet design so that $h_{k_x,k_y}(s)$ become smaller
- Introduce magnetic elements capable of creating a cancelling effect

Tune-shift and spread

First order correction to the tunes $\longrightarrow$ derivatives with respect to the action of average part of H_1. For a given term $h_{k_x,k_y}(s)x^{k_x}y^{k_y}$:

$$\delta Q_x = \frac{J_x^{k_x/2-1} J_y^{k_y/2}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

$$\delta Q_y = \frac{J_x^{k_x/2} J_y^{k_y/2-1}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

where $\bar{g}_{j,k,l,m}$ the average of $g_{j,k,l,m}(s)$ around the ring.

- Remarks
  - If $\delta Q_{x,y}$ independent of $J_{x,y}$ $\longrightarrow$ tune-shift
  - If $\delta Q_{x,y}$ depends on $J_{x,y}$ $\longrightarrow$ tune-spread (or amplitude detuning)
  - $\delta Q_{x,y} = 0$ for $k_x = j + k$ or $k_y = l + m$ odd $\longrightarrow$ go to higher order
  - Leading order tune-shift $\longrightarrow$ impact of errors and non-linear effects

The single resonance treatment

General two dimensional Hamiltonian:

$$H(\mathbf{J},\boldsymbol{\varphi}) = H_0(\mathbf{J}) + \varepsilon H_1(\mathbf{J},\boldsymbol{\varphi})$$

with the perturbed part periodic in angles:

$$H_1(\mathbf{J},\boldsymbol{\varphi}) = \sum_{k_1,k_1} H_{k_1,k_2}(J_1,J_2)\exp[i(k_1\varphi_1 + k_2\varphi_2)]$$

Resonance $n_1\omega_1 + n_2\omega_2 = 0 \longrightarrow$ blow up the solution

$$\downarrow$$

Canonical transformation **VII**: $(\mathbf{J},\boldsymbol{\varphi}) \longmapsto (\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}})$ eliminate one action:

$$F_r(\hat{\mathbf{J}},\boldsymbol{\varphi}) = (n_1\varphi_1 - n_2\varphi_2)\hat{J}_1 + \varphi_2\hat{J}_2$$

The transformed Hamiltonian

$$\hat{H}(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}}) = \hat{H}_0(\hat{\mathbf{J}}) + \varepsilon\hat{H}_1(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}})$$

ant the perturbation

$$\hat{H}_1(\hat{\mathbf{J}},\hat{\boldsymbol{\varphi}}) = \sum_{k_1,k_2} H_{k_1,k_2}(\hat{\mathbf{J}})\exp\left\{\frac{i}{n_1}\left[k_1\hat{\varphi_1} + (k_1 n_2 + k_2 n_1)\hat{\varphi_1}\right]\right\}$$

The single resonance treatment (cont.)

Relations between the variables

$$J_1 = n_1 \hat{J}_1 \quad , \qquad J_2 = \hat{J}_2 - n_2 \hat{J}_1$$
$$\hat{\varphi}_1 = n_1 \varphi_1 - n_2 \varphi_2 \quad , \qquad \hat{\varphi}_2 = \varphi_2 \quad .$$

Transformation to a rotating frame where $\dot{\hat{\varphi}}_1 = n_1 \dot{\varphi}_1 - n_2 \dot{\varphi}_2$ measures deviation from resonance.

Average over the “slow” angle $\hat{\varphi}_2 = \varphi_2$

$$\bar{H}(\hat{\mathbf{J}}, \hat{\boldsymbol{\varphi}}) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_1(\hat{\mathbf{J}}, \hat{\varphi}_1)$$

with $\bar{H}_0(\hat{\mathbf{J}}) = \hat{H}_0(\hat{\mathbf{J}})$ and
$\bar{H}_1(\hat{\mathbf{J}}, \hat{\varphi}_1) = \langle \hat{H}_1(\hat{\mathbf{J}}, \hat{\varphi}_1) \rangle_{\hat{\varphi}_2} = \sum_{p=-\infty}^{+\infty} H_{-pn_1, pn_2}(\hat{\mathbf{J}}) \exp(-ip\hat{\varphi}_1)$

The averaging eliminated one angle and thus $\hat{J}_2 = J_2 + J_1 \frac{n_2}{n_1}$ is an invariant of motion.

The single resonance treatment (cont.)

Assume dominant Fourier harmonics for $p = 0, \pm 1$

$$\bar{H}(\hat{\mathbf{J}}, \hat{\theta}_1) = \bar{H}_0(\hat{\mathbf{J}}) + \varepsilon \bar{H}_{0,0}(\hat{\mathbf{J}}) + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

Introduce $\Delta \hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$ moving reference on fixed point and expand $\bar{H}(\hat{\mathbf{J}})$ around it $\to$ Hamiltonian describing motion near a resonance:

$$\bar{H}_r(\Delta \hat{J}_1, \hat{\theta}_1) = \left. \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}} \right|_{\hat{J}_1 = \hat{J}_{10}} \frac{(\Delta \hat{J}_1)^2}{2} + 2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \cos \hat{\varphi}_1$$

Motion near a typical resonance is like that of the pendulum!!!

The libration frequency

$$\hat{\omega}_1 = \left(2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}}) \left. \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}} \right|_{\hat{J}_1 = \hat{J}_{10}} \right)^{1/2}$$

The resonance half width

$$\Delta \hat{J}_{1\ max} = 2 \left(\frac{2\varepsilon \bar{H}_{n_1,-n_2}(\hat{\mathbf{J}})}{\left. \frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}} \right|_{\hat{J}_1 = \hat{J}_{10}}} \right)^{1/2}$$

Resonance overlap criterion

(Chirikov (1960, 1979) Contopoulos (1966))

When perturbation grows $\longrightarrow$ width of resonance island grows.

$$\downarrow$$

Two resonant islands overlap $\longrightarrow$ orbits diffuse through the resonances.

Distance between two resonances

$$\delta \hat{J}_{1\ n,n'} = \frac{2\left(\frac{1}{n_1+n_2} - \frac{1}{n_1'+n_2'}\right)}{\left| \left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\right|_{\hat{J}_1=\hat{J}_{10}} \right|}$$

we get a simple resonance overlap criterion $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \delta \hat{J}_{n,n'}$

Considering width of chaotic layer and secondary islands, we have the "two thirds" rule $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \frac{2}{3} \delta \hat{J}_{n,n'}$

Limitation: geometrical nature $\longrightarrow$ difficult to extend in systems with $n \geq 3$

Single resonance theory for the accelerator Hamiltonian

(Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

The single resonance accelerator Hamiltonian

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(Q_x J_x + Q_y J_y) + g_{n_x,n_y} \frac{2}{R} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\frac{s}{R})$$

with $g_{n_x,n_y} e^{i\phi_0} = g_{j,k,l,m;p}$.

From the generating function

$$F_r(\phi_x, \phi_y, \hat{J}_x, \hat{J}_y, s) = (n_x\phi_x + n_y\phi_y - p\frac{s}{R})\hat{J}_x + \phi_y \hat{J}_y$$

we get the Hamiltonian

$$\hat{H}(\hat{J}_x, \hat{J}_y, \phi_x) = \frac{(n_x Q_x + n_y Q_y - p)\hat{J}_x + \hat{J}_y}{R} + g_{n_x,n_y} \frac{2}{R} (n_x \hat{J}_x)^{\frac{k_x}{2}} (n_y \hat{J}_x + \hat{J}_y)^{\frac{k_y}{2}} \cos(\hat{\phi_x} + \phi_0)$$

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Resonance widths

The two invariants are the new action and Hamiltonian. In the old variables:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = (Q_x - \frac{p}{n_x + n_y})J_x + (Q_y - \frac{p}{n_x + n_y})J_y$$

$$+ 2g_{n_x,n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x\phi_x + n_y\phi_y + \phi_0 - p\frac{s}{R}) \quad .$$

n_x,n_y with opposite sign (*difference* resonances) $\longrightarrow$ bounded motion
n_x, n_y with same sign (*sum* resonances) $\longrightarrow$ unbounded motion

$\star\star\star$ these are first order perturbation theory considerations $\star\star\star$

Distance from the resonance $e = n_x Q_x + n_y Q_y - p \longrightarrow$ *resonance stop band width* :

$$\Delta e = \frac{g_{n_x,n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$

The choice of the working point

During design, impose periodic structure stronger than 1

Resonance condition $n_x Q_x + n_y Q_y = p = mN$, with m the *super-periodicity*

If $p = mN \longrightarrow$ *structural* or *systematic* resonances

If $p \neq mN \longrightarrow$ *non-structural* or *random*

Major design points for high-intensity rings:

- Choose the working point far from structural resonances
- Prevent the break of the lattice super-symmetry

The choice of the working point (cont.)

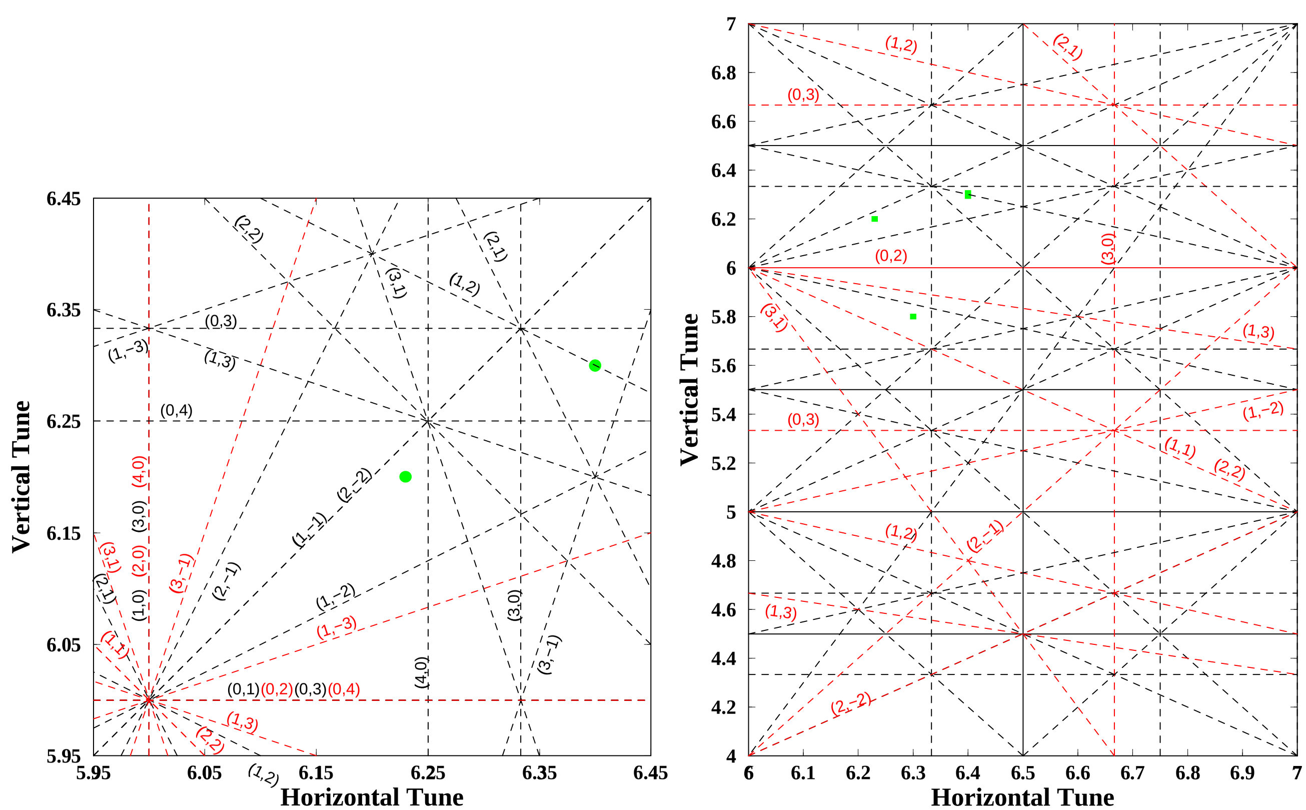

Figure 12: Tune spaces for a lattice with super-period four. The red lines are the structural resonances and the black are the non-structural (up to 4th order).

Symplectic Maps

$\mathbf{z}, \bar{\mathbf{z}}$ two sets of canonical variables

Transformation from $\mathbf{z}$ to $\bar{\mathbf{z}} \longrightarrow$ *Mapping* $\mathcal{M}$

$$\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$$

The Jacobian matrix $M = M(\mathbf{z}, t)$ is $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$

$\mathcal{M}$ *Symplectic* if $M^T \ J \ M = J$ where $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$

$$\mathcal{M} \text{ symplectic} \longrightarrow [\bar{z}_i, \bar{z}_j] = [z_i, z_j] = J_{ij}$$

$$\downarrow$$

Symplectic maps preserve the Poisson brackets

Lie Formalism

Poisson brackets properties:

$$[af + bg, h] = a[f, h] + b[g, h] \;, a, b \in \mathbb{R}$$

$$[f, g] = -[g, f]$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

$$[f, gh] = [f, g]h + g[f, h]$$

Set of functions of $(\mathbf{q}, \mathbf{p}, \mathbf{t}) \longrightarrow$ linear vector space

$$\downarrow$$

Poisson bracket *Lie Algebra*

Lie Operator $:f:$

$$:f:g = [f, g]$$

and $:f:^2 g = [f, [f, g]]$, etc.

Lie Formalism (cont.)

For a Hamiltonian system $H(\mathbf{z}, t)$:

$$\frac{d\mathbf{z}}{dt} = [H, \mathbf{z}] =: H : \mathbf{z}$$

and the solution $\mathbf{z}(t)$ with $\mathbf{z}(0) = \mathbf{z}_0$ can be obtained formally as:

$$\mathbf{z}(t) = \sum_{k=0}^{\infty} \frac{t^k : H :^k}{k!} \mathbf{z}_0 = e^{t:H:} \mathbf{z}_0$$

with $\mathcal{M} = e^{:H:}$ a symplectic map.

The Campbell-Baker-Hausdorff theorem ensures that

$$e^{sA} e^{tB} = e^{C}$$

A, B and C real matrixes and s,t sufficiently small.

In an accelerator the one turn map $\mathcal{M}$ mapping the origin into itself can be written as:

$$\mathcal{M} = e^{:f_2:} \, e^{:f_3:} \, e^{:f_4:} \, \ldots$$

where f_m homogeneous polynomials of degree m in $z_1, \ldots, z_n$.

Normal Form Construction

The one turn map: $\mathbf{z}' = \mathcal{M}(\mathbf{z})$ where $\mathbf{z} = (x, p_x, y, p_y) \in \mathbb{R}^4$

$$\begin{array}{ccc} \mathbf{z} & \xrightarrow{M} & \mathbf{z}' \\ \Phi^{-1}\downarrow & & \downarrow\Phi^{-1} \\ \boldsymbol{u} & \xrightarrow[U]{} & \boldsymbol{u}' \end{array}$$

with $\mathcal{M} = \Phi^{-1} \circ U \circ \Phi$ and $U = e^{:H:}$ and $\Phi = e^{:F:}$. The transformation in the new set of variables is

$$\boldsymbol{\zeta} = e^{-:F_r:}\ \mathbf{h}$$

where $h^{\pm}_{x,y} = \sqrt{2J_{x,y}}\ e^{\mp i\phi_{x,y}}$ is the resonance basis and $\zeta^{\pm}_{x,y}(N) = \sqrt{2I_{x,y}}\ e^{\mp i\psi_{x,y}(N)}$ with $\psi_{x,y}(N) = 2\pi N\nu_{x,y} + \psi_{x,y_0}$. We have:

$$F_r = \sum_{jklm} f_{jklm} \zeta_x^{+\,j} \zeta_x^{-\,k} \zeta_y^{+\,l} \zeta_y^{-\,m} \quad \text{with} \quad j,k,l,m \in \mathbb{Z}$$

and thus

$$F_r = \sum_{jklm} f_{jklm} (2I_x)^{\frac{j+k}{2}} (2I_y)^{\frac{l+m}{2}} e^{-i\psi_{jklm}}$$

Graphical Resonance Representation

$$F_r \approx \sum_{jklm} f_{jklm} \, \epsilon_x^{\frac{j+k}{2}} \, \epsilon_y^{\frac{l+m}{2}} \, e^{-i\psi_{jklm}}$$

where the emittances are $\epsilon_x = n_\sigma \sqrt{\frac{\epsilon_n}{\gamma}} \cos K$ and $\epsilon_y = n_\sigma \sqrt{\frac{\epsilon_n}{\gamma}} \sin K$ For a resonance $a\nu_x + bq_y = c$ we have:

$$F_{(a,b)} \approx \sum_{\substack{jklm \\ j+k+l+m \leq n \\ j+k=a \, , \, l+m=b}} f_{jklm} \, \epsilon_x^{\frac{a}{2}} \, \epsilon_y^{\frac{b}{2}}$$

where we took the contribution of the resonance terms up to an order n and arbitrarily set $\psi_{jklm} = 0$.

Graphical Resonance Representation (cont.)

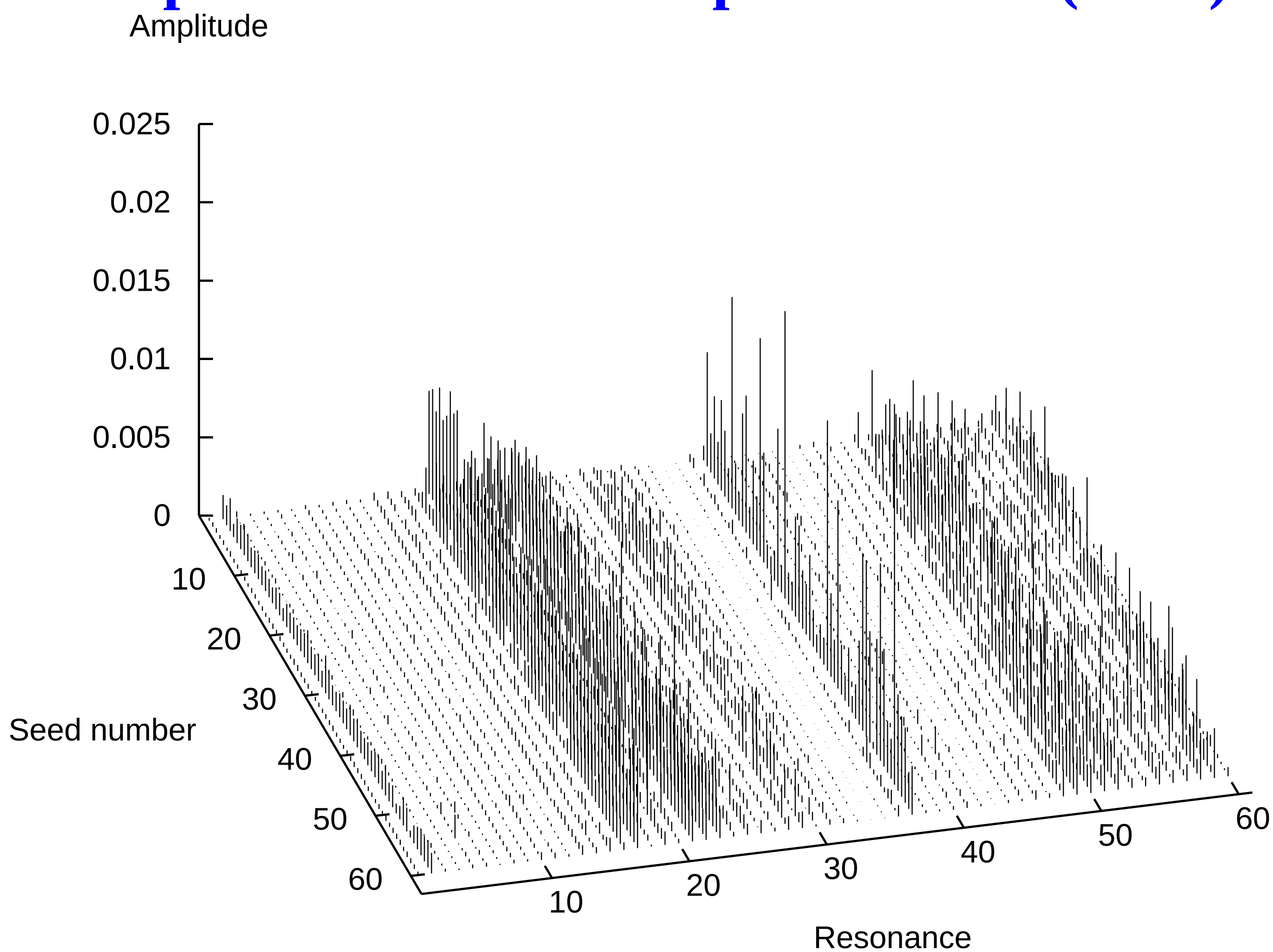

Figure 13: Norm of the 7th order resonance coefficients f_{jklm} of th generating function, for 60 seeds of LHC optics versions 5 using the "nominal" error table.

Graphical Resonance Representation (cont.)

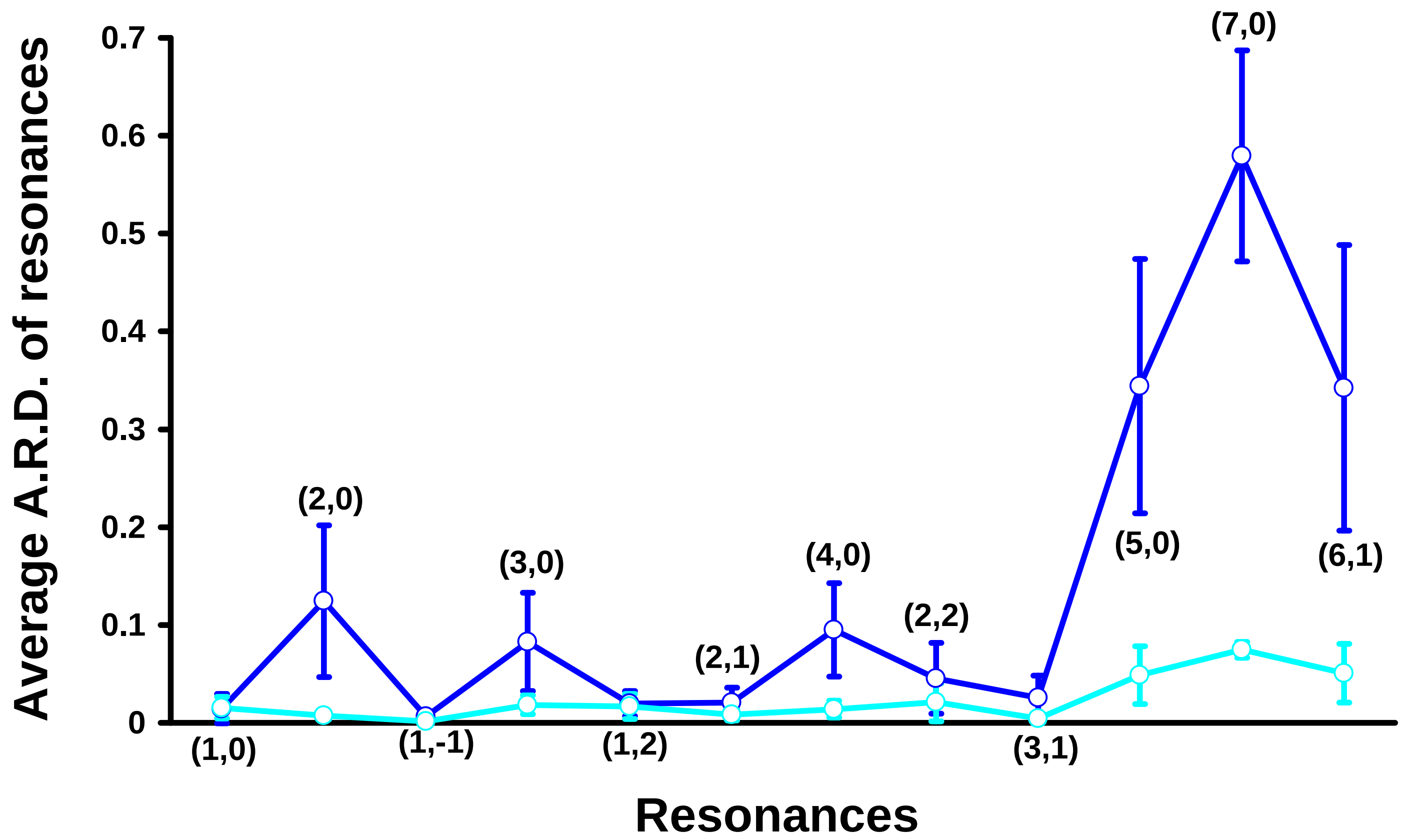

Figure 14: Average absolute value of the resonance strengths relative difference at 8σ and 15° between LHC optics version 4 and 5 with and without the errors on the “warm” quadrupoles.

Frequency Maps

Perturbation theory $\longrightarrow$ insight about non-linear

Problem: Construction of normal forms or action variables cannot be applied for large perturbations

Need a method to represent the systems' global dynamics

$\downarrow$

Frequency map analysis (Laskar 1988, 1990)

Algorithm to precisely compute frequencies associated with KAM tori of tracked orbits

Frequency Maps (cont.)

NAFF algorithm $\longrightarrow$ quasi-periodic approximation, truncated to order N,

$$f_j'(t) = \sum_{k=1}^{N} a_{j,k} e^{i\omega_{jk} t} \ ,$$

with $f_j'(t), a_{j,k} \in \mathbb{C}$ and $j = 1, \ldots, n$, of a complex function $f_j(t) = q_j(t) + ip_j(t)$, formed by a pair of conjugate variables determined by usual numerical integration, for a finite time span $t = \tau$.

Recover the frequency vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \longrightarrow$ parameterizes KAM tori.

Frequency Maps (cont.)

Construct frequency map by repeating procedure for set of initial conditions.

Example: Keep all the $\boldsymbol{q}$ variables constant, and explore the momenta $\boldsymbol{p}$ to produce the map $\mathcal{F}_\tau$:

$$\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{p}\big|_{\boldsymbol{q}=\boldsymbol{q}_0} & \longrightarrow & \boldsymbol{\nu} \; . \end{array}$$

Dynamics of the system analyzed by studying the regularity of the frequency map.

Frequency Maps (cont.)

F.M.A is applied to tracking data N,

$$f'_j(t) = \sum_{k=1}^{N} a_{j,k} e^{i\omega_{jk} t} ,$$

with $f'_j(t), a_{j,k} \in \mathbb{C}$ and $j = 1, \ldots, n$, of a complex function $f_j(t) = q_j(t) + ip_j(t)$, formed by a pair of conjugate variables determined by usual numerical integration, for a finite time span $t = \tau$.

Recover the frequency vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \longrightarrow$ parameterizes KAM tori.

Frequency Maps (cont.)

Construct frequency map by repeating procedure for set of initial conditions.

Example: Keep all the $\boldsymbol{q}$ variables constant, and explore the momenta $\boldsymbol{p}$ to produce the map $\mathcal{F}_\tau$:

$$\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \boldsymbol{p}|_{\boldsymbol{q}=\boldsymbol{q}_0} & \longrightarrow & \boldsymbol{\nu} \; . \end{array}$$

Dynamics of the system analyzed by studying the regularity of the frequency map.

Frequency Maps (cont.)

F.M.A is applied to tracking data ($\tau = 500$ turns), for large number of initial conditions ($\approx 10^4$).

Particle coordinates distributed uniformly on Courant-Snyder invariants A_{x0} and A_{y0}, at different ratios A_{x0}/A_{y0}.

The map is

$$\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (I_x, I_y)|_{p_x, p_y = 0}, & \longrightarrow & (\nu_x, \nu_y) \end{array},$$

Frequency Maps for the LHC

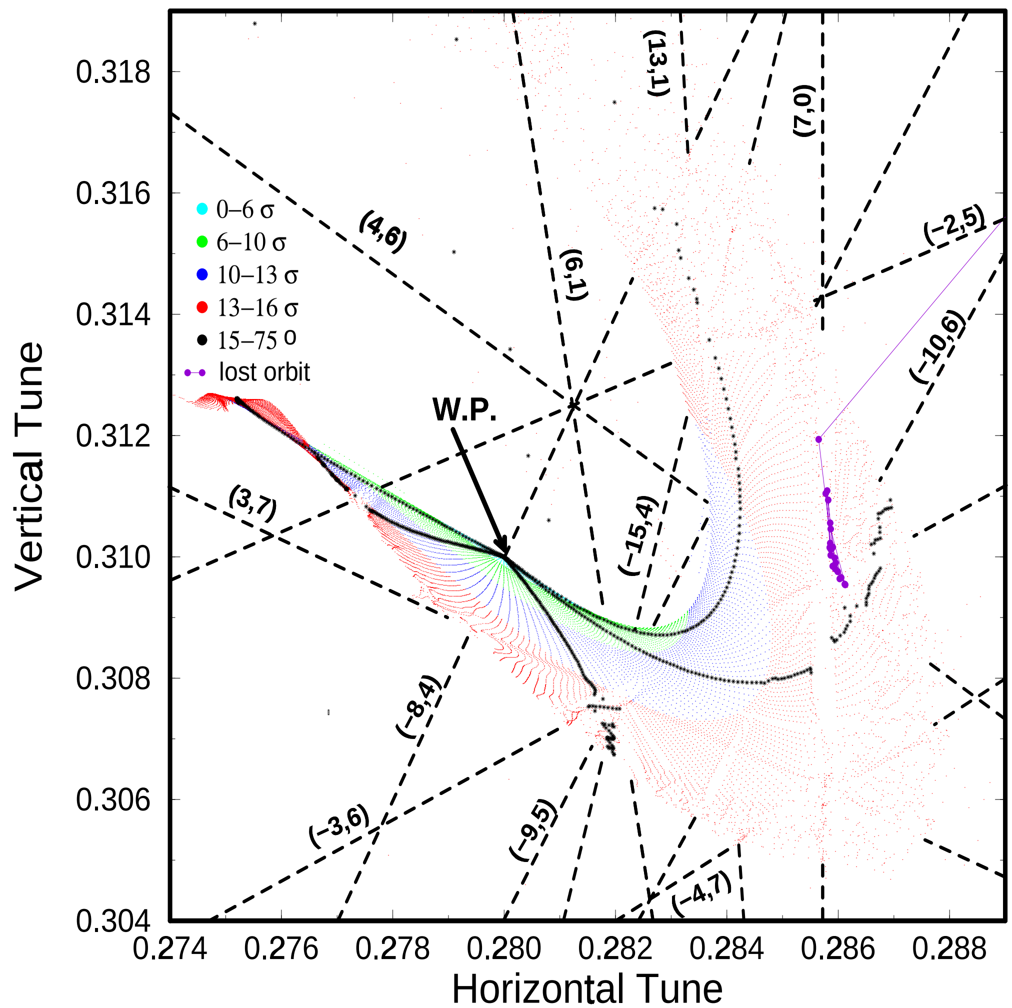

Figure 15: Frequency map for the LHC optics version 5 using the target error table.

Frequency Maps for the LHC (cont.)

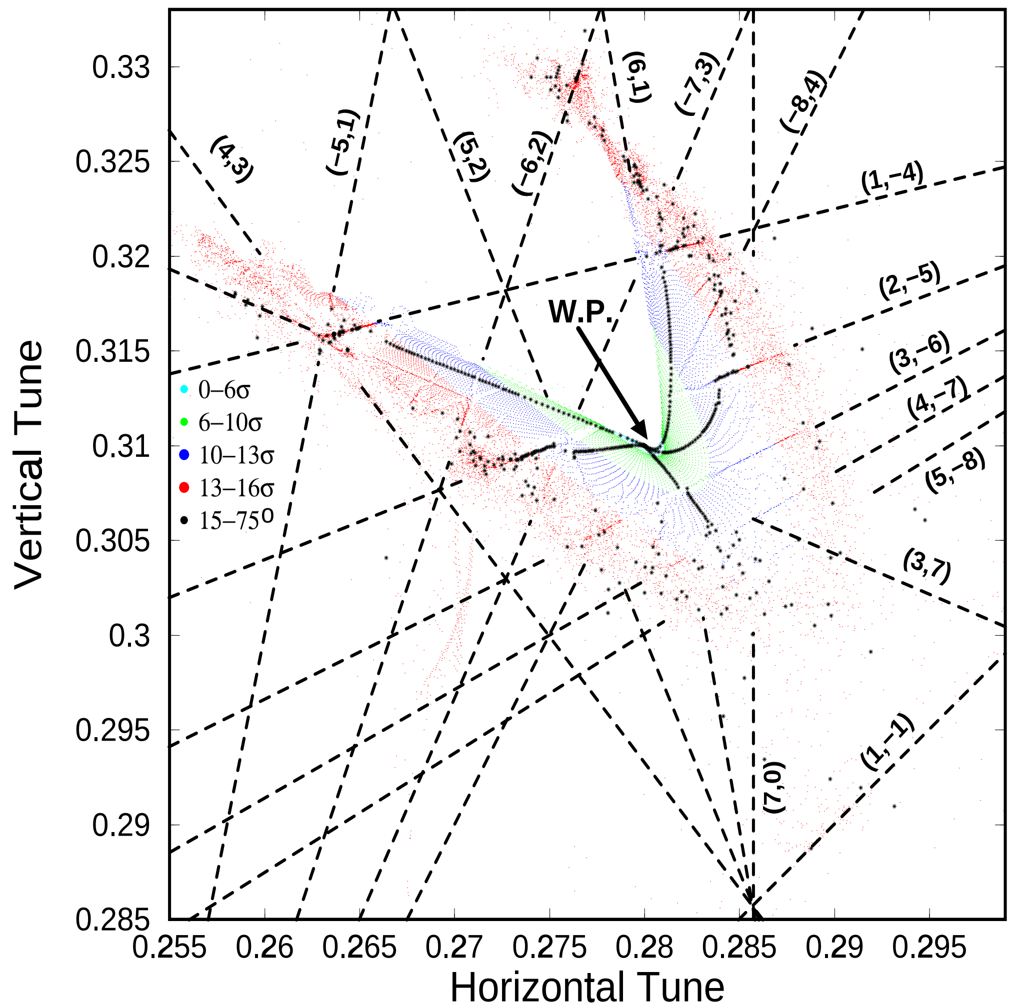

Figure 16: Frequency map for the LHC optics version 5 with a big skew octupole error in the main dipoles.

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Diffusion Maps

Calculate tune for two equal and successive time spans and compute diffusion vector:

$$\boldsymbol{D}|_{t=\tau} = \boldsymbol{\nu}|_{t\in(0,\tau/2]} - \boldsymbol{\nu}|_{t\in(\tau/2,\tau]} \ ,$$

Plot the points in (A_{x0}, A_{y0})–space with a different colors

- grey for stable ($|\boldsymbol{D}| \leq 10^{-7}$) to
- black for strongly chaotic particles ($|\boldsymbol{D}| > 10^{-2}$).

Diffusion quality factor:

$$D_{QF} = \langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \rangle_R \ .$$

Diffusion Maps for the LHC (cont.)

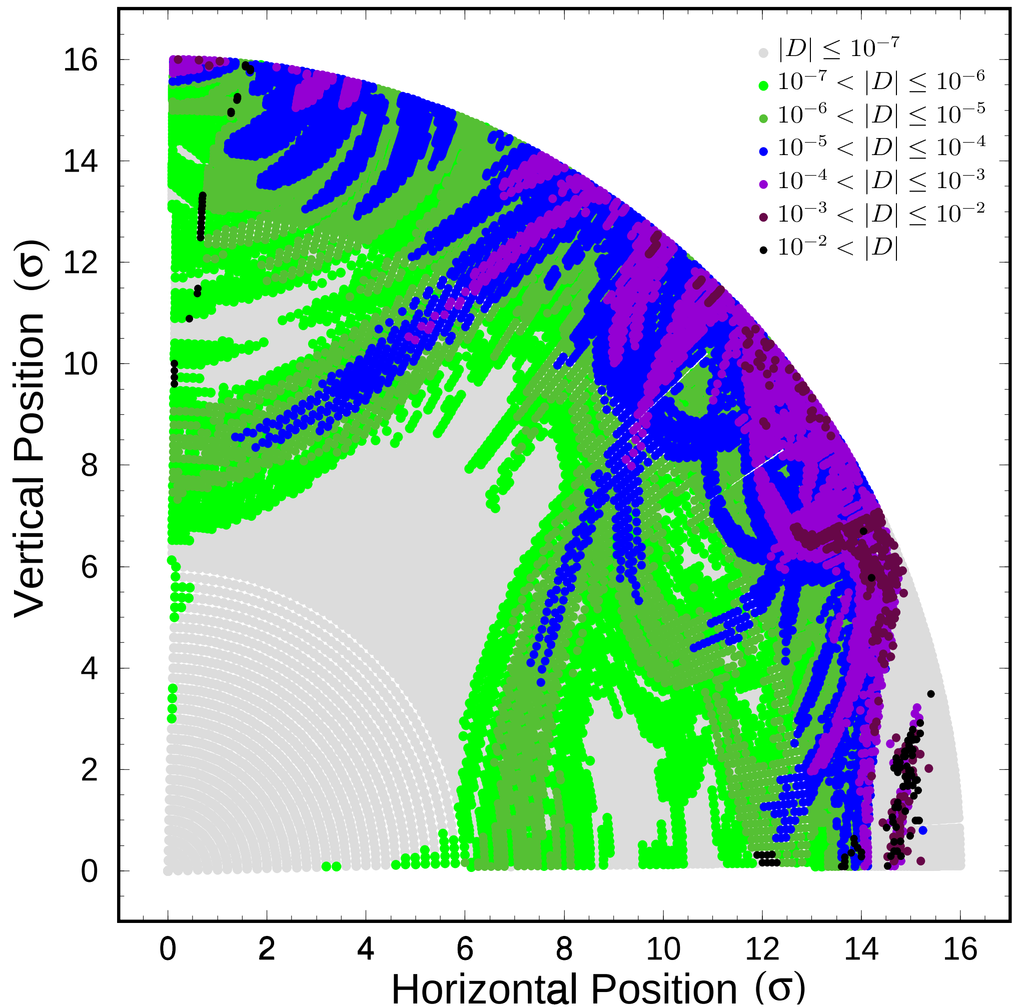


Figure 17: Diffusion maps for the LHC optics version 5 using the target error table.

Diffusion Maps for the LHC (cont.)

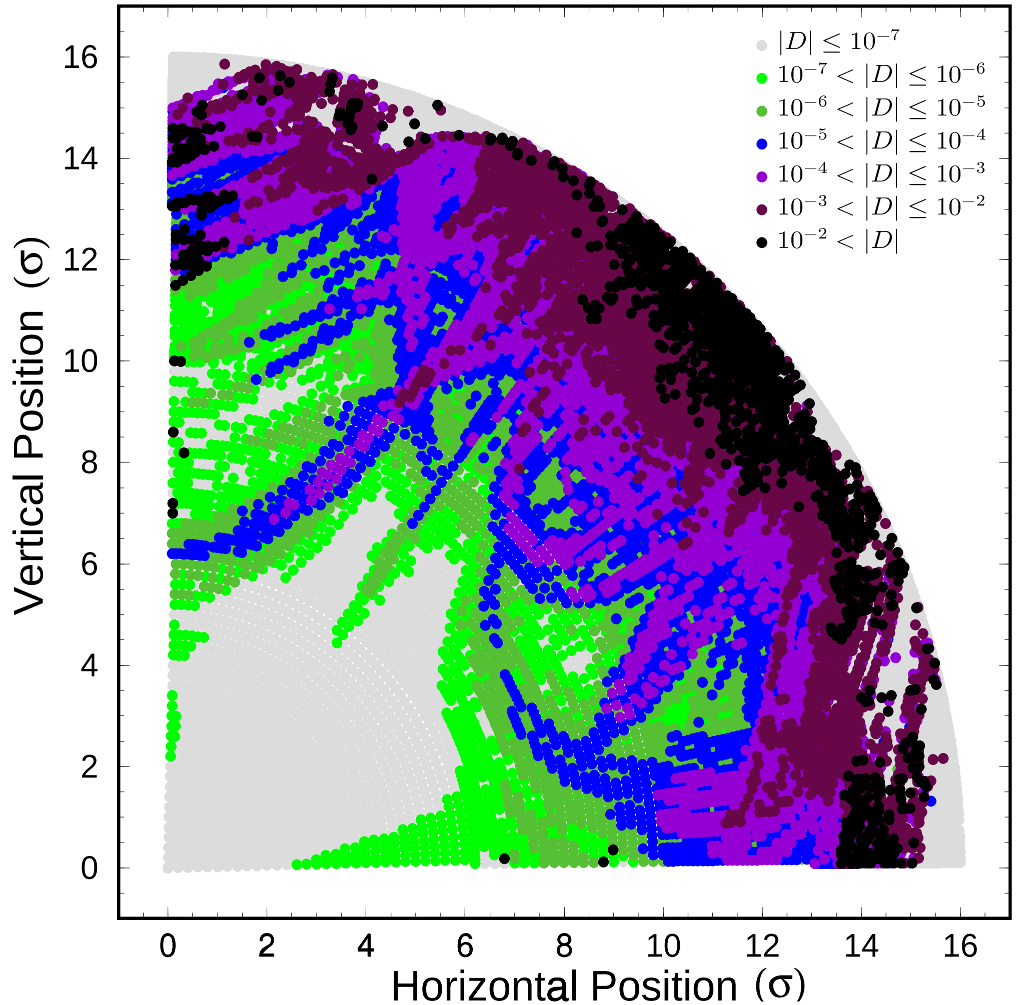


Figure 18: Diffusion maps for the LHC optics version 5 using the target error table.

Choosing the best working point for the SNS ring

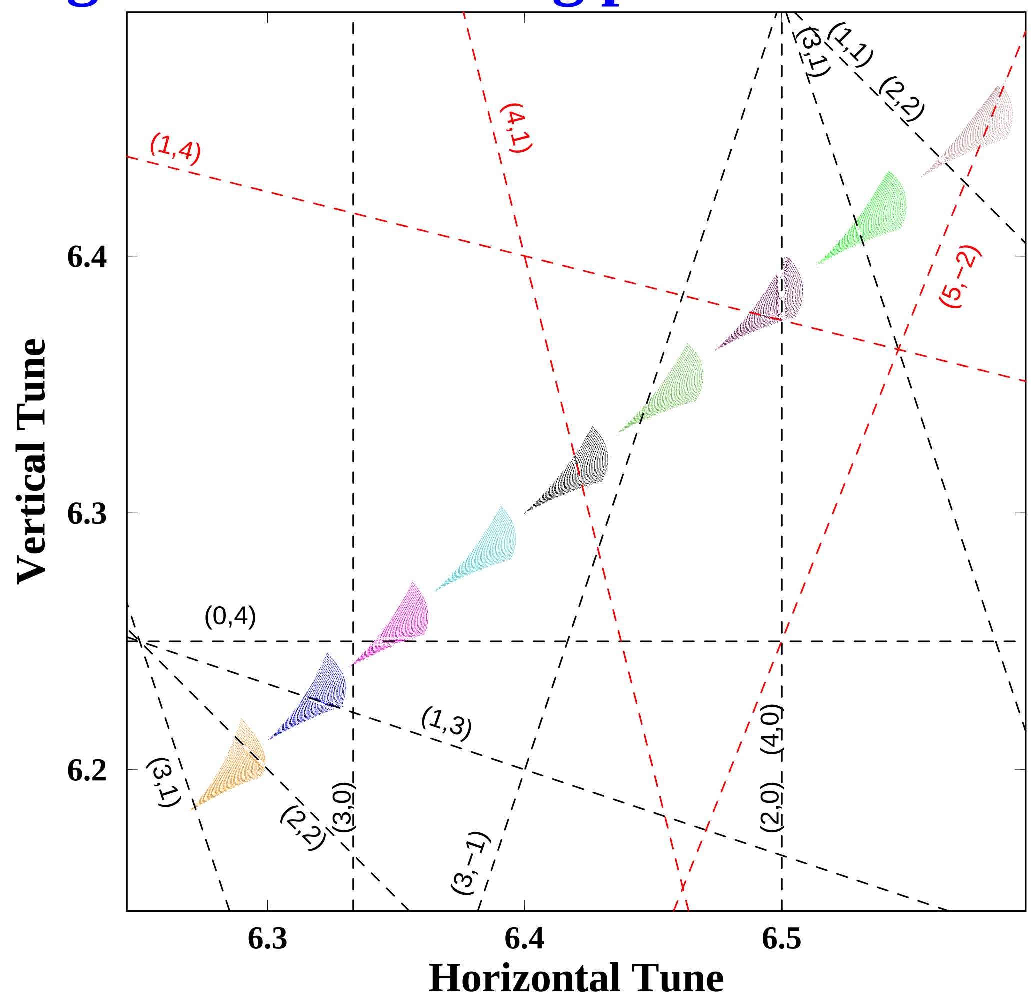


Figure 19: Frequency maps for the working point (6.4,6.3), for 9 different $\delta p/p$, from +2% (left bottom corner) to -2% (right upper corner)

Choosing the best working point for the SNS ring (cont.)

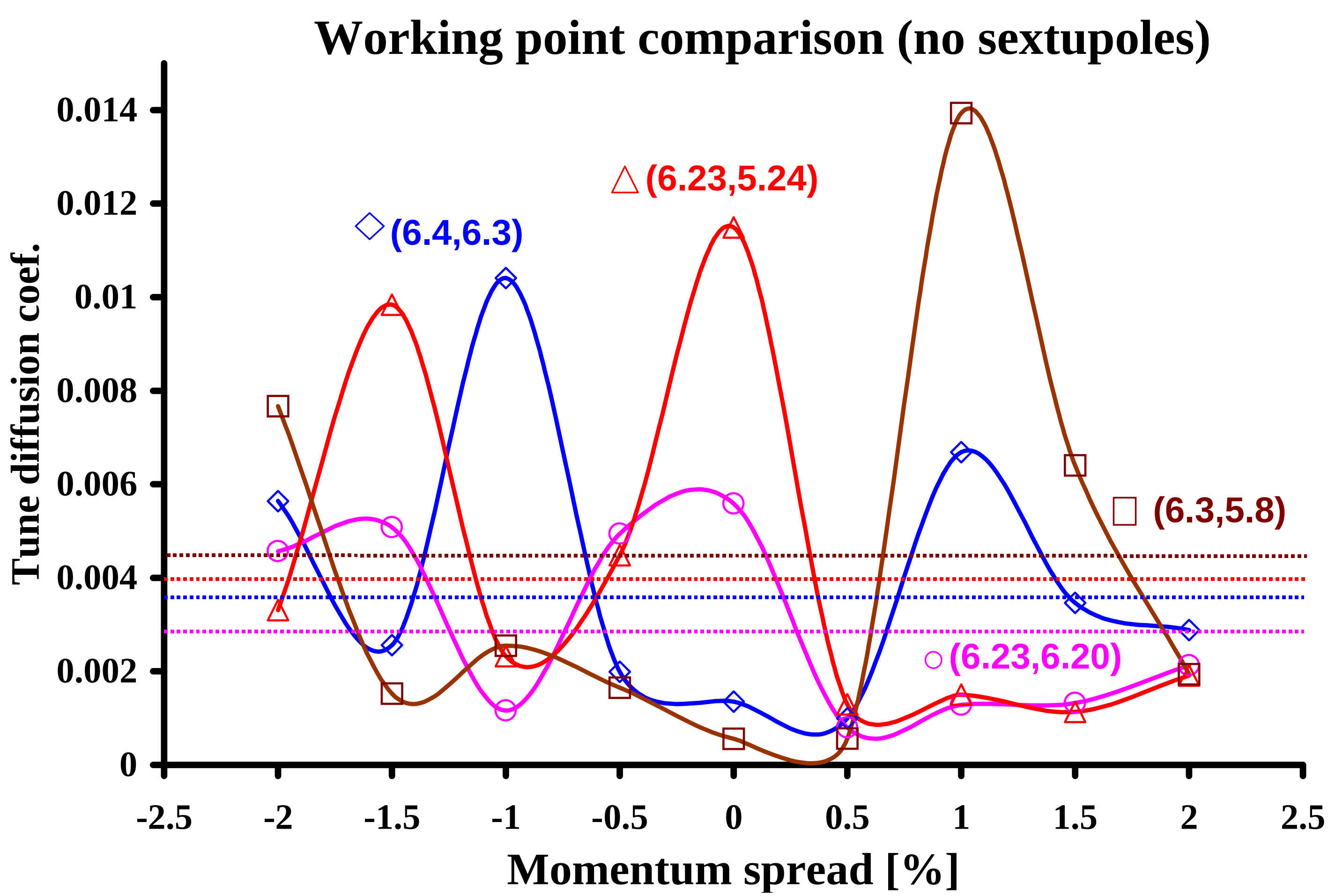
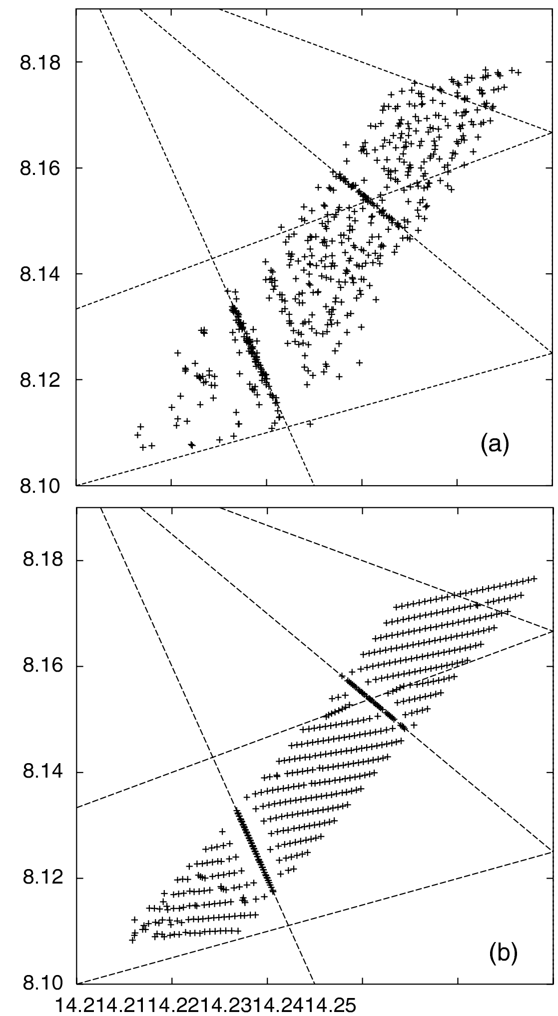


Figure 20: Tune diffusion coefficients for all SNS working points versus the different momentum spreads, for natural chromaticity. The dashed lines are the average of the coefficient over all momentum spreads.

Experimental frequency maps for the ALS

Robin et al. (2000)

- Horizontal and vertical pinger magnets to kick the beam in both directions and in one turn (rise time of 600ns)
- Bunch train of 4×10^{10} electrons (10 mA)
- Record 1024 turns on a BPM synchronized with the kick (1/20 of dumping time)
- 600 kicks (20 secs per kick or 4 hours)



Tune-shift and resonance driving terms in the SPS

Bartolini et al. (1999)

- Coasting beam of 10^{12} protons @ 120 GeV
- $Q_x = 26.637$, $Q_y = 26.533$ with corrected chromaticity
- Closed orbit, coupling corrected, without dumpers and octupoles
- Extraction sextupoles on
- Horizontal kicks with fast extraction kicker
- Data of 170 turns recorded on pick-ups