Special Relativity and Electromagnetism

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Outline

- Notions of Special Relativity
  - Historical background
  - Lorentz transformations (length contraction and time dilatation)
  - 4-vectors and Einstein's relation
  - Conservation laws, particle collisions
- Electromagnetic theory
  - Maxwell's equations,
  - Magnetic vector and electric scalar potential
  - Lorentz force
  - Electromagnetic waves

Historical background

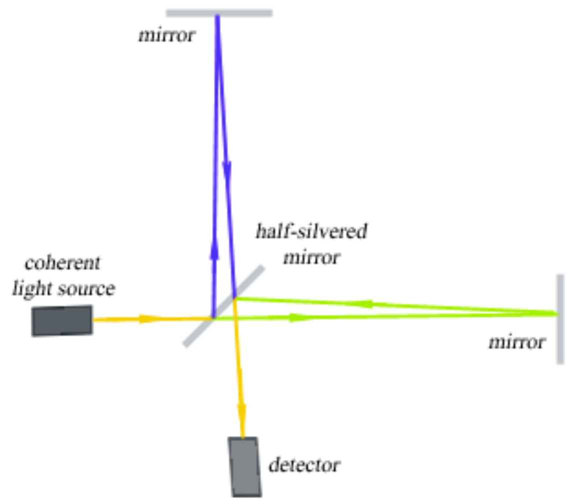
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - A few doubtful hypotheses and inconsistencies
    - A medium called "luminiferous ether" exists for the transport of electromagnetic waves
    - "Ether" has a small interaction with matter and is carried along with astronomical objects
    - Light propagates with speed $c = 3\times10^8$ m/s in " but not invariant in all frames
    - Maxwell's equations are not invariant under Galilean transformations
    - In order to make electromagnetism compatible with classical mechanics, assume that light has speed *c* only in frames where source is at rest

Experimental evidence

- Star light aberration: a small shift in apparent positions of distant stars due to the finite speed of light and
- Fizeau and Foucault's experiments (1850) on the velocity of light in air and liquids
- Michelson-Morley experiment (1887) to detect motion of the earth through ether
- Lorentz-FitzGerland contraction hypothesis (1894 – 1904): perhaps bodies get compressed in the direction of their motion by a factor

Fizeau and Foucault

Michelson



$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Postulates of Special Relativity

1. First Postulate: **Principle of Relativity**
   Every **physical law is invariant** under inertial co-ordinate transformations. Thus, if an object in space-time obeys the mathematical equations describing a physical law in one inertial frame of reference, it must necessarily obey the same equations when using any other inertial frame of reference.
1. Second Postulate: **Invariance of the speed of light**
   There exists an **absolute constant** $0 < c < \infty$ with the following property. If *A*, *B* are two events which have co-ordinates (t, x_1, x_2, x_3) and (s, y_1, y_2, y_3) in one inertial frame *F*, and have co-ordinates (t', x'_1, x'_2, x'_3) and (s', y'_1, y'_2, y'_3) in another inertial frame *F*', then

$$\sqrt{(x'_1 - y'_1)^2 + (x'_2 - y'_2)^2 + (x'_3 - y'_3)^2} = c(s' - t')$$

if and only if

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = c(s - t)$$

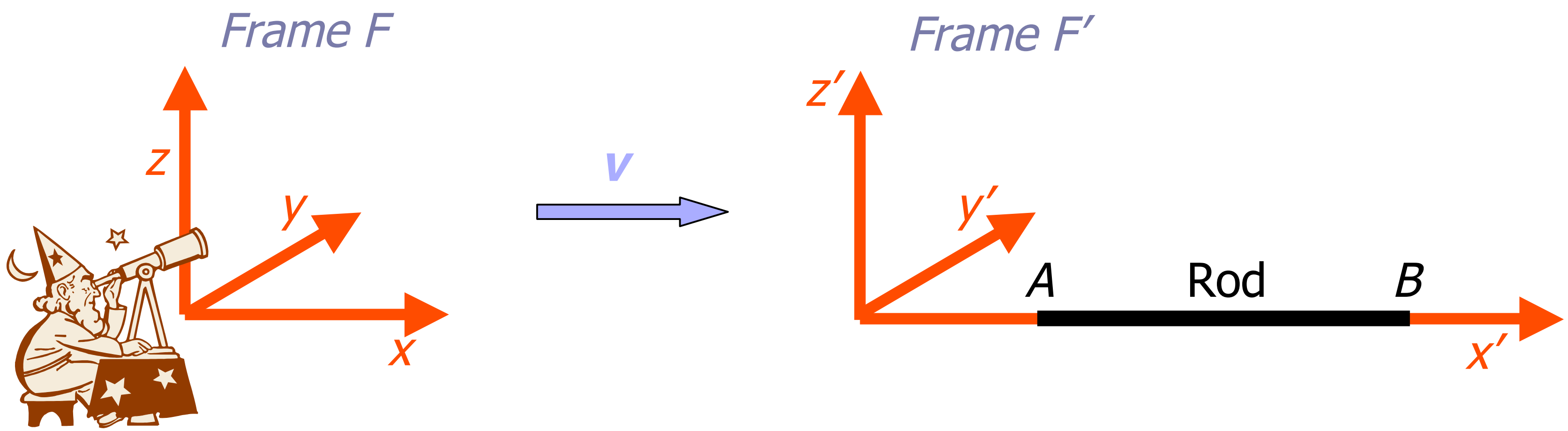
The Lorentz Transformation

- From a frame $F(t,x,y,z)$ to a frame $F'(t',x',y',z')$ moving with velocity v along the x-axis the space-time coordinates are transformed as:

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \text{with} \begin{cases} \beta = \dfrac{v}{c} \\ \gamma = \dfrac{1}{\sqrt{(1-\beta^2)}} \end{cases}$$

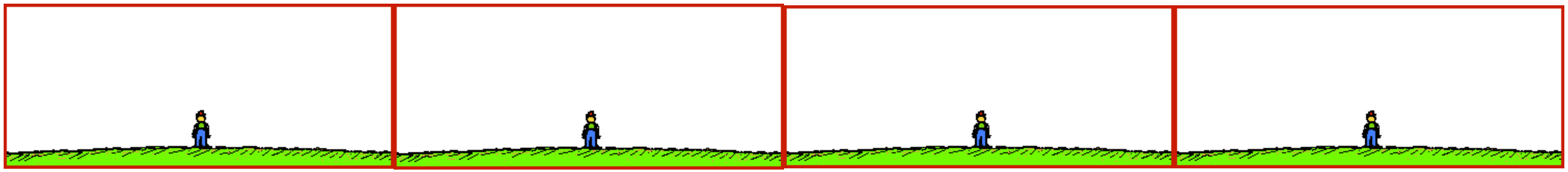
- The space-time interval is invariant under Lorentz transformations

$$ds = dx^2 + dy^2 + dz^2 - c\,t^2$$



Rod *AB* of *length L′* fixed in *F′* at x'_A, x'_B. Its *length L* seen by the observer is **contracted**

$$L' = x'_B - x'_A = \gamma(x_B - x_A) = \gamma L > L$$

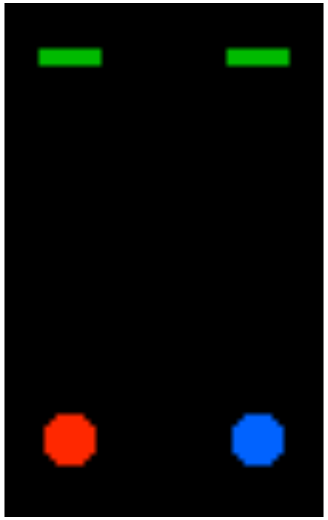


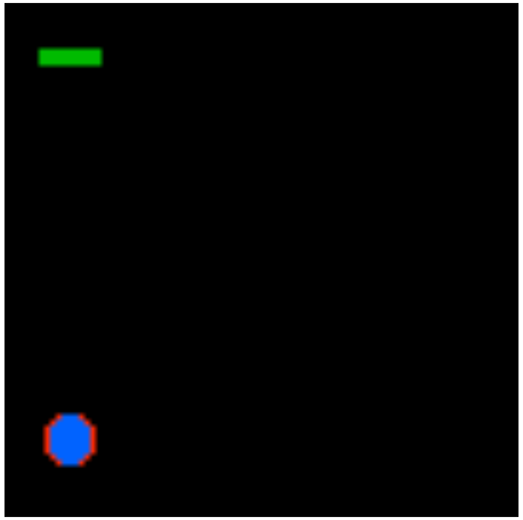
$v = 0.1\ c$ $v = 0.865\ c$ $v = 0.99\ c$ $v = 0.9999\ c$

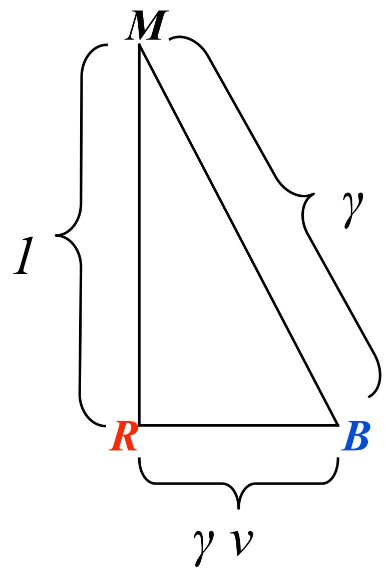
Time dilatation

Clock in frame F at a point with coordinates (x,y,z) and different times t_A and t_B.In moving frame F' the *time difference* $\Delta t'$ is **dilated**

$$\Delta t' = t'_B - t'_A = \gamma(t_B - t_A) = \gamma \Delta t > \Delta t$$







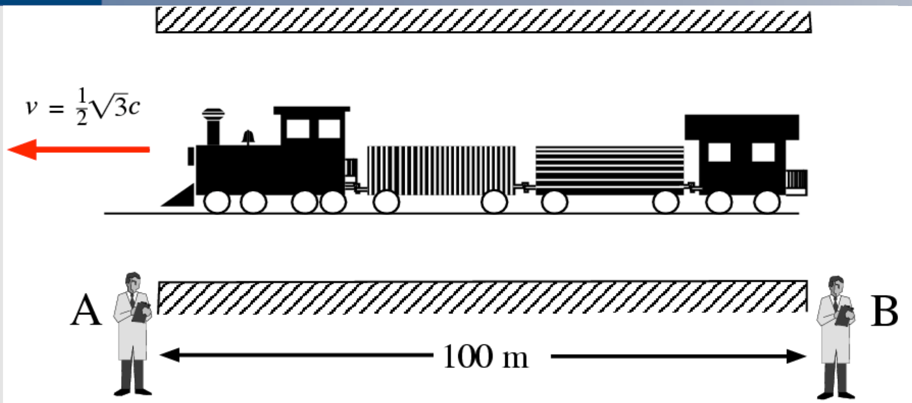
Red and **Blue** with identical clocks, (light beam bouncing off mirror). When **Red** and **Blue** at rest, tick and tocks are simultaneous

Blue moves with his mirror at velocity v. **Blue** measures the same time between ticks and tocks but according to **Red**, the clock of **Blue** runs slow

For $c = 1$, distance between mirror and **Red** $RM = 1$ tick. **Red** thinks that distance between **Blue** and mirror is $BM = \gamma$ ticks. Distance between **Blue** and **Red** $RB = \gamma v$. So,

$$\gamma^2 = (\gamma v)^2 + 1$$

Example: Relativistic Train



- F frame of tunnel with observers at both ends A, B

$$x_A = 0\ ,\quad x_B = 100\ \text{m}$$

- Relativistic train on frame F′ with driver D and guard G

$$x'_D = 0\ ,\quad x'_G = 100\ \text{m}$$

- Observers A and B see train contracted

$$L' = (x'_G - x'_D) = \gamma(x_G - x_D) = \gamma L \Rightarrow$$
$$L = (x'_G - x'_D)/\gamma = 50\ \text{m}$$

- Tunnel moves relative to train and the D,G see tunnel of 50 m!

$$S = (x_A - x_B) = \gamma(x'_A - x'_B) = \gamma S' \Rightarrow$$
$$S' = (x_A - x_B)/\gamma = 50\ \text{m}$$

- What does B′s clock read when G goes into tunnel? The two events (t_B, x_B), (t'_G, x'_G) are coincident and

$$x'_G = \gamma(x_B + vt_B) \Rightarrow t_B = -\frac{100}{c\sqrt{3}}$$

- What does G′s clock read as he enters?

$$x_B = \gamma(x'_G - vt'_G) \Rightarrow t'_G = +\frac{100}{c\sqrt{3}}$$

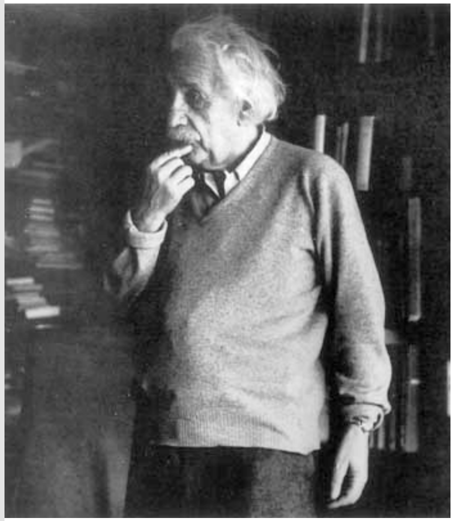
- Where is the guard G, with respect to A, when his clock reads 0? Setting the time to zero in the previous equation

$$x_{G,0} = \gamma x'_G \Rightarrow x_{G,0} = 200m$$

So the guard is still 100m from tunnel entrance!!!

4-Vectors and Einstein's relation

- First note that $\frac{dt}{d\tau} = \gamma$ and $m = \gamma m_0$.
- 4-Position: $\mathbf{X} = (c\,t, \mathbf{x})$
- 4-Velocity: $\mathbf{V} = \frac{d\mathbf{X}}{d\tau} = \gamma\,\frac{d\mathbf{X}}{dt} = \gamma\,\frac{d}{dt}(c\,t, \mathbf{x}) = \gamma\,(c, \mathbf{v})$
- 4-Acceleration: $\mathbf{A} = \frac{d\mathbf{V}}{d\tau} = \gamma\,\frac{d\mathbf{V}}{dt} = \gamma\,\frac{d}{dt}(c\,t, \mathbf{v}) = \gamma\,(\dot{\gamma}c, (\dot{\gamma\mathbf{v}}))$
- 4-Momentum: $\mathbf{P} = m_0\mathbf{V} = m_0\gamma\,(c, \mathbf{v}) = (m\,c, m\mathbf{v}) = (m\,c, \mathbf{p})$
- 4-Force: $\mathbf{F} = m_0\mathbf{A} = \gamma(m_0\dot{\gamma}c, m_0(\dot{\gamma\mathbf{v}})) = \gamma(\dot{m}c, \mathbf{f})$
- Invariants: $\mathbf{V}\cdot\mathbf{V} = c^2$ and $\mathbf{P}\cdot\mathbf{P} = m_0^2c^2$



Differentiating the momentum invariant $\mathbf{V}\cdot\frac{d\mathbf{P}}{d\tau} = 0$ and $\mathbf{V}\cdot\mathbf{F} = 0$

i.e. which gives $\dot{m}c^2 - \mathbf{v}\cdot\mathbf{f} = 0$. But the rate of change of the kinetic energy is $\frac{dT}{dt} = \mathbf{v}\cdot\mathbf{f} = \dot{m}c^2$ and by integrating $T = mc^2 + \text{constant}$.

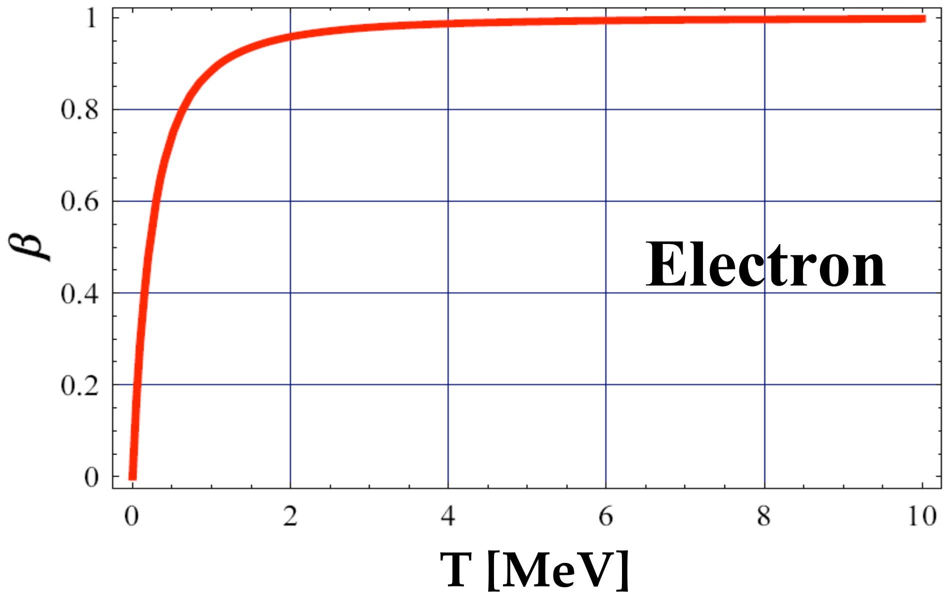
At rest $T = 0$, and thus the constant is $-m_0c^2$. Finally, the energy is

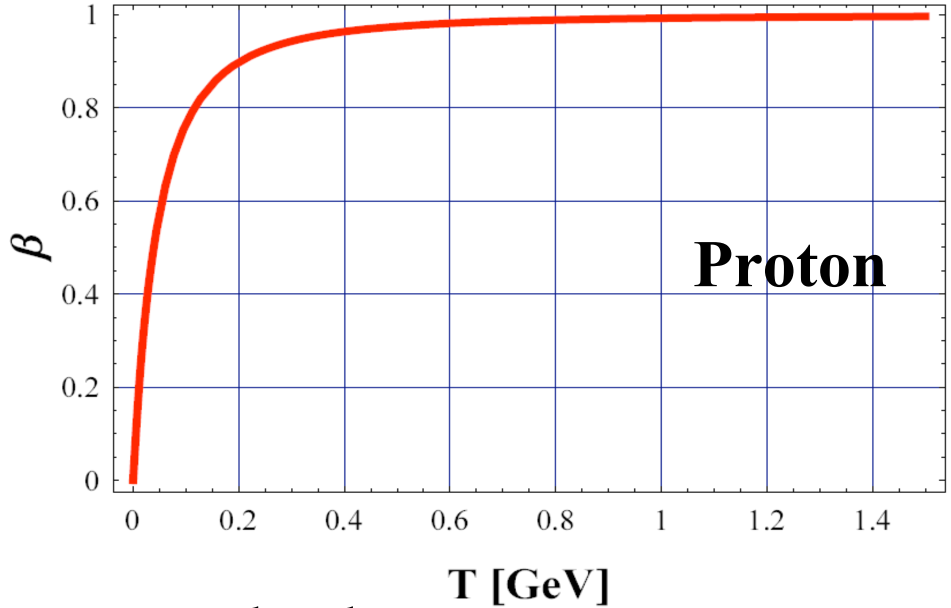
$$E \equiv T + E_0 = T + m_0c^2 = mc^2$$

Velocity and Kinetic Energy

- Relative velocity $\beta = \frac{v}{c}$
- Relative velocity and Lorentz factor $\beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$
- Momentum $p = mv = \gamma m_0 v = \beta\gamma m_0 c$
- Kinetic energy $T = (m - m_0)c^2 = m_0 c^2(\gamma - 1)$

For $v << c$, $\gamma \approx 1 + \frac{v^2}{2c^2}$ and $T = \frac{1}{2} m_0 v^2$





- Heavy particles become relativistic at higher energies
- When relativistic, small velocity change provides big change in energy

Relationships for small parameter variations

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$ $\frac{\Delta\gamma}{\gamma} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$ $(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$

Momentum conservation laws

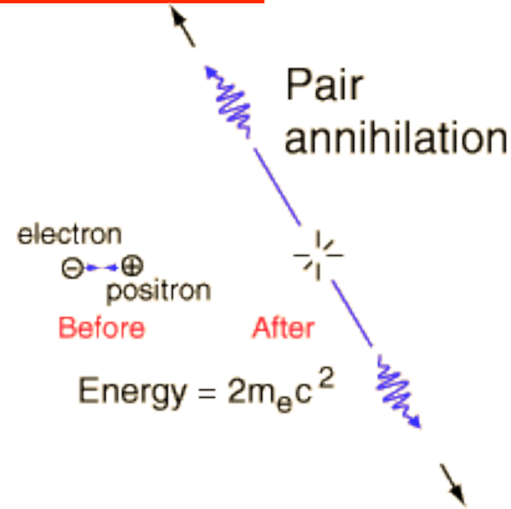
- Equivalent expression for 4-momentum

$$\mathbf{P} = m_0 \mathbf{V} = m_0 \gamma\ (c, \mathbf{v}) = (m\, c, \mathbf{p}) = (\frac{E}{c}, \mathbf{p})$$

- From the momentum invariance we get $$\frac{E^2}{c^2} = m_0^2 c^2 + \mathbf{p}^2$$
- As the 4-momentum is conserved

$$\sum_{i,\text{particles}} \mathbf{P_i} = \text{constant}$$

  - Total energy is conserved $\sum_{i,\text{particles}} E_i = \text{constant}$
  - Classical momentum is conserved $\sum_{i,\text{particles}} \mathbf{p_i} = \text{constant}$



  - Norm of the 4-momentum is conserved $\sum_{i,\text{particles}} |\mathbf{P_i}| = \text{constant}$

- Momentum is conserved but mass is not (mass is a form of energy)!!!

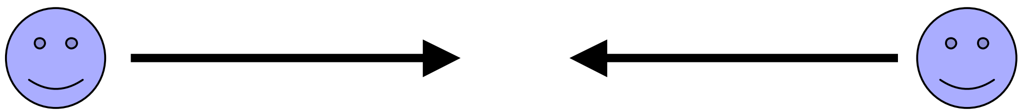
Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E_{lab}.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Mass Frame (CMF): Velocities are equal and opposite, total energy is E_{cm}.



$$\mathbf{P_1} = (E_{\text{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\text{cm}}/(2c), -\mathbf{p})$$

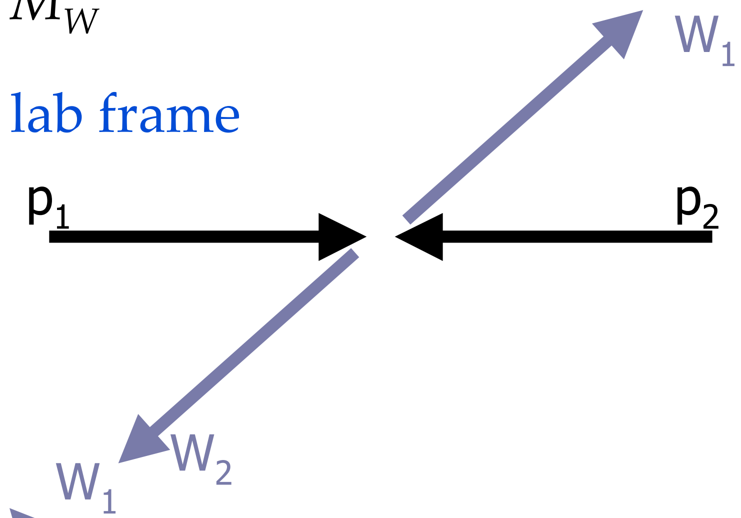
- The quantity $(\mathbf{P_1} + \mathbf{P_2})^2$ is invariant.
- In the **CMF**, we have $(\mathbf{P_1} + \mathbf{P_2})^2 = E_{\text{cm}}^2/c^2$.
- In general $(\mathbf{P_1} + \mathbf{P_2})^2 = \mathbf{P_1}^2 + \mathbf{P_2}^2 + 2\mathbf{P_1} \cdot \mathbf{P_2} = 2m_0^2c^2 + 2\mathbf{P_1} \cdot \mathbf{P_2}$
- In the **LF**, we have $\mathbf{P_1} \cdot \mathbf{P_2} = E_1 m_0$ and $(\mathbf{P_1} + \mathbf{P_2})^2 = 2m_0 E_{\text{lab}}$.

Finally $E_{\text{cm}}^2 = 2m_0 c^2 E_{\text{lab}}$

Particle on target vs. colliding particles

A proton p_1 collides with an anti-proton p_2 (same rest mass m_0), producing two particles W_1 and W_2 with mass M_W

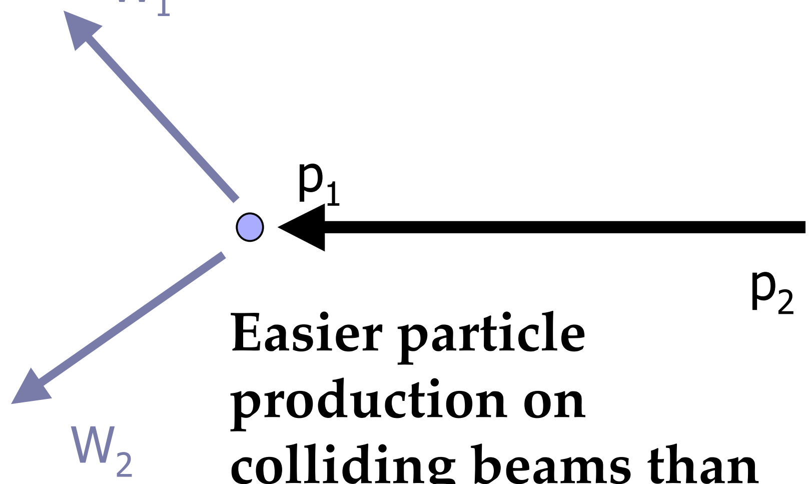
1. p_1 , p_2 with equal and opposite velocities in lab frame



$$(\mathbf{P_1} + \mathbf{P_2})^2 = E_{\rm cm}^2/c^2 = E_{\rm W}^2/c^2 \quad \text{and}$$

$$E_{\rm cm} = E_{\rm W} \geq 2M_W c^2 \approx \boxed{180\ \text{GeV}}$$

2. p_1 with 0 velocity in lab frame.



$$E_{\rm cm}^2 = 2m_p c^2 E_{\rm lab}$$

$$E_{\rm lab} > \frac{2M_W^2 c^2}{m_p} \text{ and as } E = E_{\rm lab} - m_p c^2$$

we have $E \geq (\frac{2M_W^2 c^2}{m_p} - m_p)c^2 \approx \boxed{1.36 \times 10^4\ \text{GeV}}$

Easier particle production on colliding beams than fixed target experiments

Maxwell's Equations

- Accelerator physics: description of charged particle dynamics in the presence of electromagnetic fields
- Maxwell's equations relate Electric and Magnetic fields generated by charge and current distributions.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss law: divergence of the electric field gives the density of the sources.

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Faraday's law of induction: induced electric field in coil equal to negative rate of change of magnetic field

$$\nabla \cdot \mathbf{B} = 0$$

Gauss law for magnetism: there are no magnetic monopoles

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$$

Ampere-Maxwell's law: integral of magnetic field in closed loop proportional to current flowing in the loop (static electric field)

$\boldsymbol{E}$ = electric field	[V/m]	μ_0 (permeability of free space) = 4π 10^{-7} [C $V^{-1}m^{-1}$]
$\boldsymbol{B}$ = magnetic flux density	[T]	ε_0 (permittivity of free space) = 8.854 10^{-12} [V s $A^{-1}m^{-1}$]
ρ = charge density	[C/m^3]	c (speed of light) = 2.99792458 10^8 m/s
$\boldsymbol{j}$ = current density	[A/m^2]	$1/c^2 = \varepsilon_0\, \mu_0$

Vector and scalar potential

- Maxwell's equation is a set of coupled first order differential equations relating different components of E/M field
- Introduce potentials to reduce number of equations and unknowns
- From Gauss law of magnetism, we obtain the **magnetic vector potential**

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- From Faraday's law, we obtain an **electric scalar potential**

$$\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \quad \rightarrow \quad \exists \Phi : \quad \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \cdot \Phi$$

- Inserting these equations back to Gauss and Ampere's law

$$\nabla^2 \Phi + \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{j}$$

Considering the **Lorentz gauge invariants**

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \Lambda$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

and choose $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

to get the decoupled equations

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}$$

Lorentz force

- Force on charged particle moving in an electromagnetic field:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- In accelerator physics: electric fields used for particle acceleration and magnetic fields for particle guidance (but not exclusively)
  - Integrate Lorentz force with the path length to get kinetic energy

$$\Delta T = \int \mathbf{F} ds = q \int \mathbf{E} ds + q \int \cancel{(\mathbf{v} \times \mathbf{B})\mathbf{v}} dt$$

  - Kinetic energy is changed from the presence of electric but not magnetic field
- Relativistic equation of motion

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m_0 \gamma \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electromagnetic waves

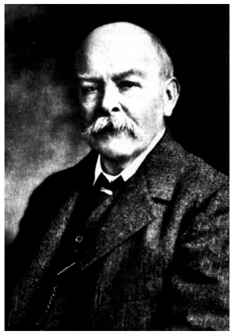
- Maxwell's equations predict the existence of electromagnetic waves, later discovered by Hertz.

- Assume no charges and no currents:
  - Take the curl of Faraday's law and replace curl of magnetic field by using Ampere's law

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (-\frac{\partial}{\partial t}\mathbf{B}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

  - Use the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$ and Gauss law to obtain a 3D wave equation

$$\nabla^2\mathbf{E} = \frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} + \frac{\partial^2\mathbf{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$



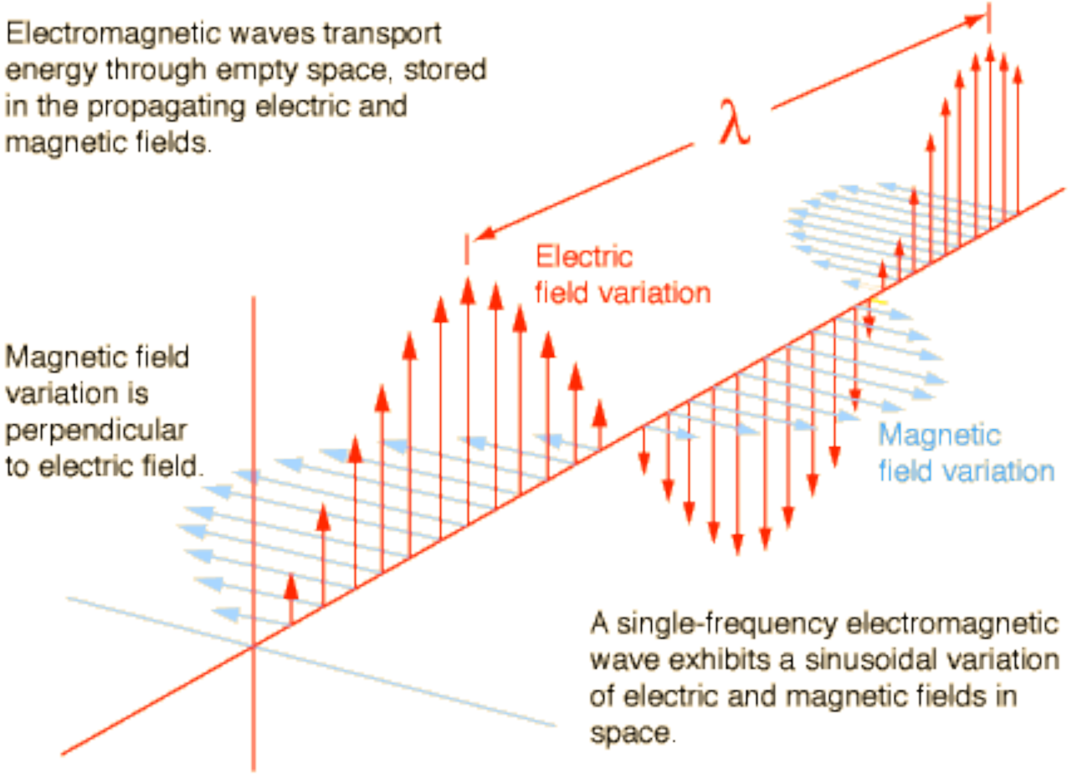
- E/M waves carry energy, with a flow (power per unit area) described by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

Nature of electromagnetic waves

- Plane wave with angular frequency ω travelling in the direction of the wave vector **k**
- From Gauss' laws
  $$\mathbf{k}\cdot\mathbf{E} = \mathbf{k}\cdot\mathbf{B} = 0$$
  i.e. fields are transverse to each other and wave propagation
- From Faraday's law
  $$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}$$
- From Ampere-Maxwell's law
  $$\mathbf{k}\times\mathbf{B} = -\omega/c^2\mathbf{E}$$
- We have that the velocity of propagation in vacuum is
  $$\frac{|\mathbf{E}|}{|\mathbf{B}|} = \frac{\omega}{|\mathbf{k}|} = c$$
  with wavelength $\lambda = \dfrac{2\pi}{|\mathbf{k}|}$
  and frequency $\nu = \dfrac{\omega}{2\pi}$

$$\mathbf{E} = \mathbf{E_0}\exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})]$$
$$\mathbf{B} = \mathbf{B_0}\exp[i(\omega t - \mathbf{k}\cdot\mathbf{x})]$$



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