Transverse Motion

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Outline - part I

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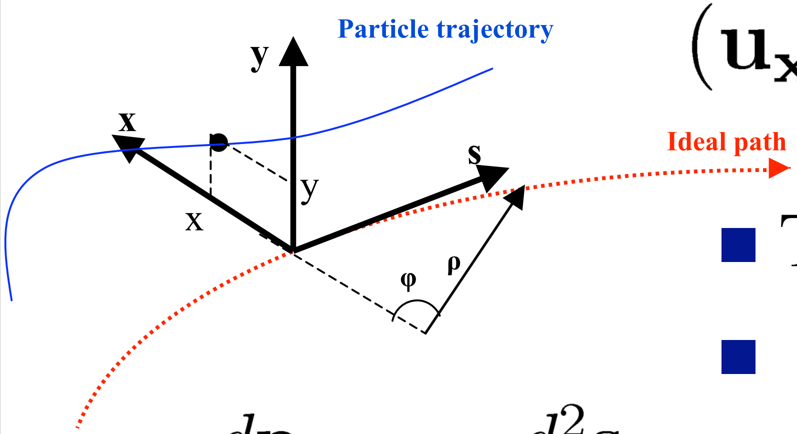
Coordinate system

- Cartesian coordinates not useful to describe motion in an accelerator
- Instead a system following an ideal path along the accelerator is used (**Frenet** reference system)



$$(\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_z}) \rightarrow (\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_s})$$

- The curvature vector is $\mathbf{k} = -\dfrac{d^2\mathbf{s}}{ds^2}$
- From Lorentz equation

$$\frac{d\mathbf{p}}{dt} = m\gamma \frac{d^2\mathbf{s}}{dt^2} = m\gamma v_s^2 \frac{d^2\mathbf{s}}{ds^2} = -m\gamma v_s^2 \mathbf{k} = q|\mathbf{v} \times \mathbf{B}|$$

- The ideal path is defined by $\mathbf{k} = -\dfrac{q}{p}|\dfrac{\mathbf{v}}{v_s} \times \mathbf{B}|$

Beam guidance

- Consider uniform magnetic field ***B*** in the direction perpendicular to particle motion. From the ideal trajectory and after considering that the transverse velocities $v_x << v_s, v_y << v_s,$ the radius of curvature is

$$\frac{1}{\rho} = |k| = |\frac{q}{p}B| = |\frac{q}{\beta E_{tot}}B|$$

- The **cyclotron** or **Larmor frequency** $\omega_L = |\frac{qc^2}{E_{tot}}B|$

- We define the **magnetic rigidity** $|B\rho| = \frac{p}{q}$

- In more practical units $\boxed{\beta E_{tot}[GeV] = 0.2998|B\rho|[Tm]}$

- For ions with charge multiplicity ***Z*** and atomic number ***A***, the energy per nucleon is

$$\beta \bar{E}_{tot}[GeV/u] = 0.2998\frac{Z}{A}|B\rho|[Tm]$$

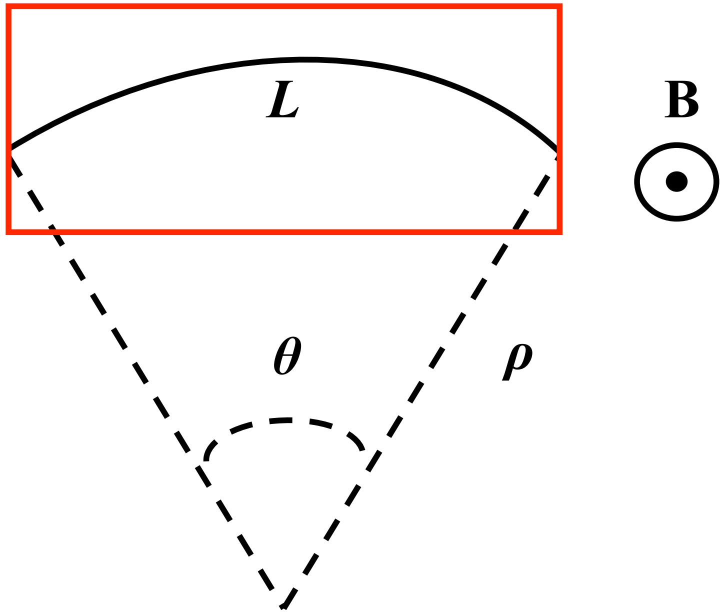
Dipoles

- Consider an accelerator ring for particles with energy ***E*** with ***N*** dipoles of length ***L***


SNS ring dipole

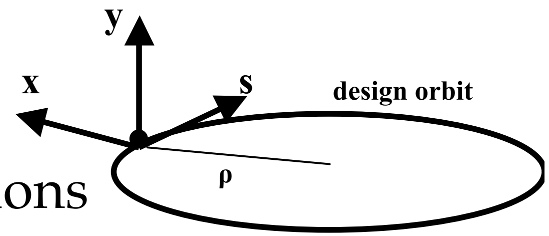
- **Bending angle** $\theta = \frac{2\pi}{N}$
- **Bending radius** $\rho = \frac{L}{\theta}$
- **Integrated dipole strength**

$$BL = \frac{2\pi}{N}\frac{\beta E}{q}$$



- Comments:
  - By choosing a dipole field, the dipole length is imposed and vice versa
  - The higher the field, shorter or smaller number of dipoles can be used
  - Ring circumference (cost) is influenced by the field choice

Beam focusing

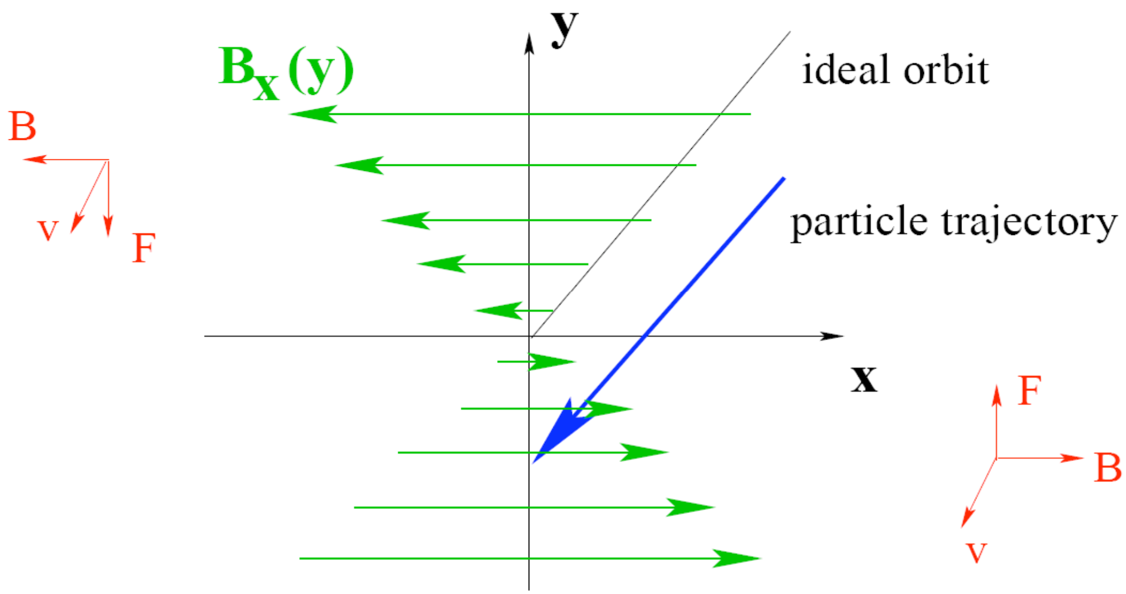


- Consider a particle in the design orbit.
- In the **horizontal plane**, it performs harmonic oscillations

$x = x_0 \cos(\omega t + \phi)$ with frequency $\omega = \frac{v_s}{\rho}$

- The horizontal acceleration is described by $\frac{d^2x}{ds^2} = \frac{d^2x}{v_s^2 dt^2} = -\frac{1}{\rho^2}x$
- There is a **week focusing** effect in the horizontal plane.
- In the **vertical plane**, the only force present is gravitation. Particles are displaced vertically following the usual law $\Delta y = \frac{1}{2} a_g \Delta t^2$
- Setting $a_g = 10\ \mathrm{m/s^2}$, the particle is displaced by **18mm** (LHC dipole aperture) in **60ms** (a few hundreds of turns in LHC)

Need of **focusing**!



Focusing elements

- Magnetic element that deflects the beam by an angle proportional to the distance from its centre (equivalent to **ray optics**) provides focusing.
- The deflection angle is defined as $\alpha = -\frac{y}{f}$, for a lens of focal length *f* and small displacements y.
- A magnetic element with length *l* and gradient *g* provides field $B_x = gy$ so that the deflection angle is

$$\alpha = -\frac{l}{\rho} = -\frac{q}{\beta E} B_x l = -\overbrace{\frac{q}{\beta E} g l}^{f^{-1}} y$$

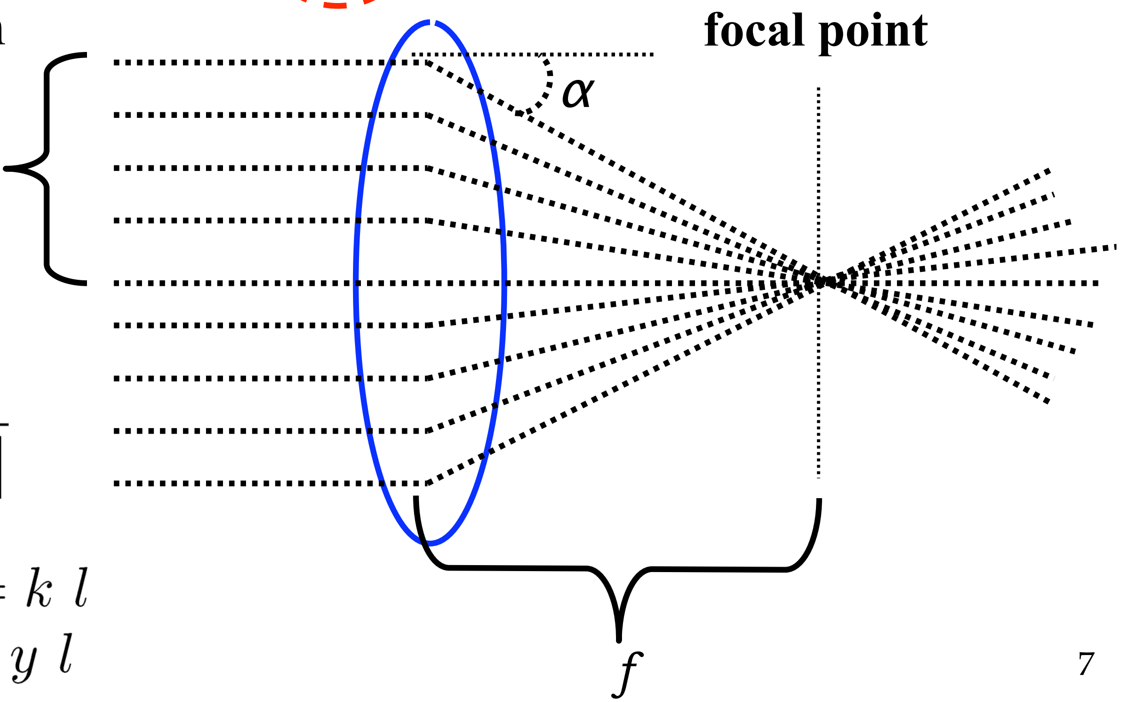
- The normalised focusing strength is defined as

$$k = \frac{qg}{\beta E}$$

- In more practical units, for ***Z***=1

$$k[m^{-2}] = 0.2998 \frac{g[T/m]}{\beta E[GeV]}$$

- The focal length becomes $f^{-1} = k\ l$ and the deflection angle is $\alpha = -k\ y\ l$

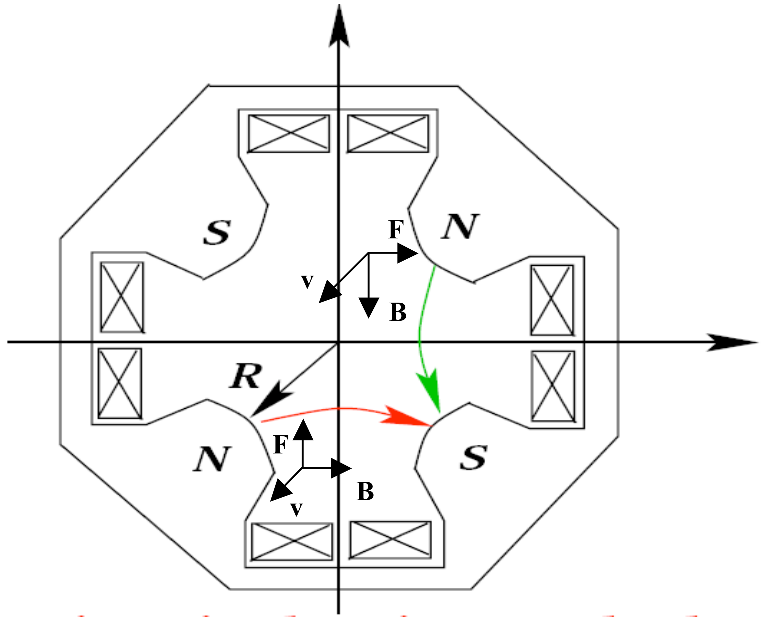


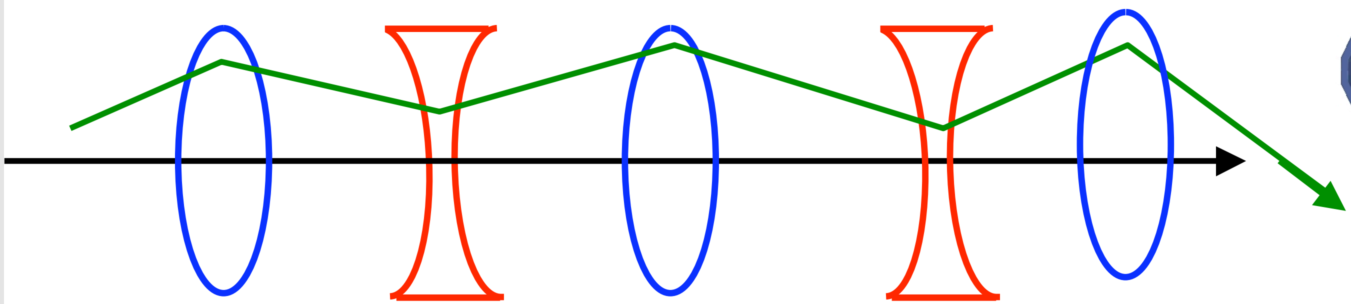
Quadrupoles

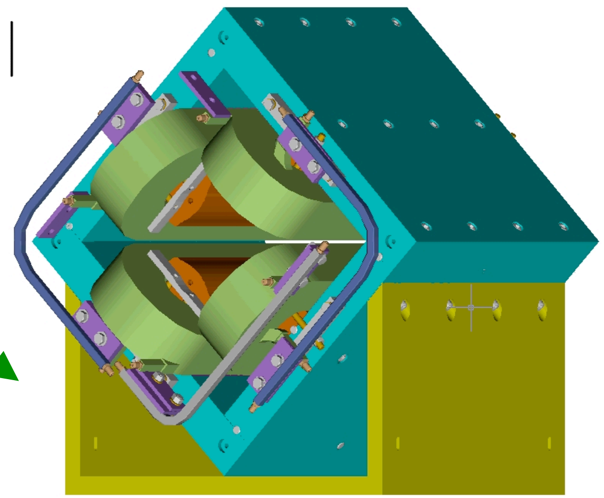
- Quadrupoles are focusing in one plane and defocusing in the other
- The field is $(B_x, B_y) = g(y, x)$
- The resulting force $(F_x, F_y) = k(y, -x)$
- Need to alternate focusing and defocusing in order to control the beam, i.e. **alternating gradient focusing**
- From optics we know that a combination of two lenses with focal lengths f_1 and f_2 separated by a distance d

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 \, f_2}$$

- If $f_1 = -f_2$, there is a **net focusing effect**, i.e. $\frac{1}{f} = \left|\frac{d}{f_1 \, f_2}\right|$







Magnetic multipole expansion

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming a 2D field in x and y, the vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe)

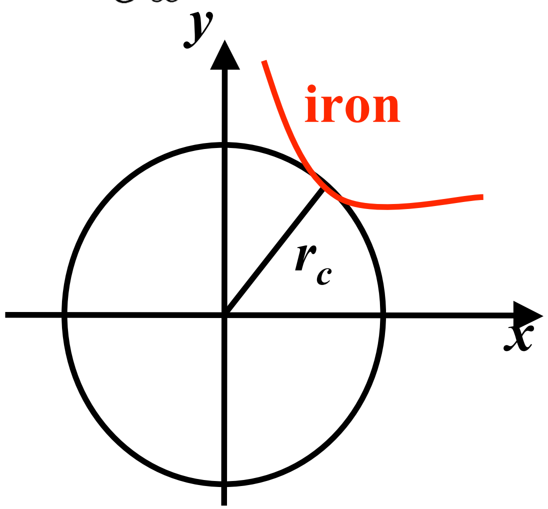
$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y} , \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function

There exist a complex potential of $z = x+iy$ with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)



$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \ , \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \ , \ a_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

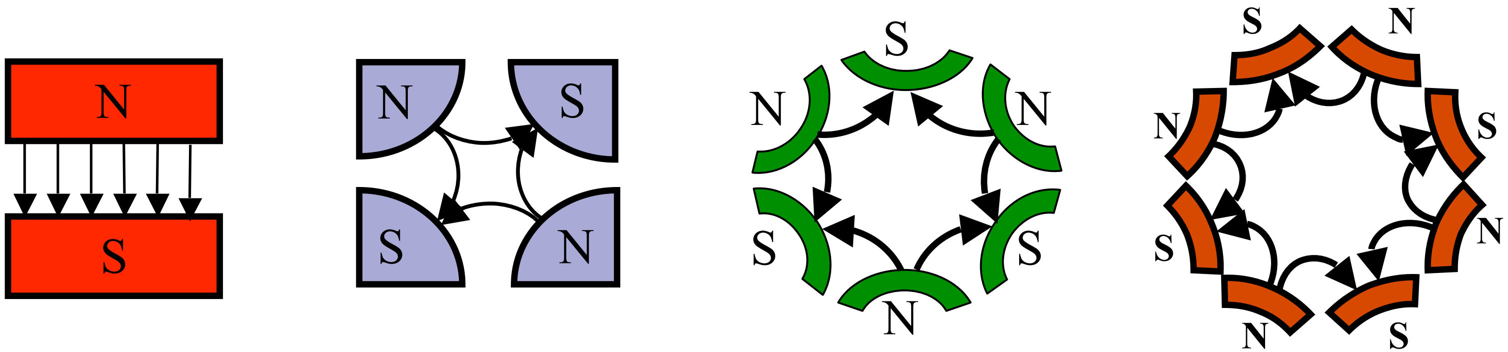
$$B_y + iB_x = 10^{-4} B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)(\frac{x+iy}{r_0})^{n-1}$$

- **Note**: *n′=n-1* is the US convention

Magnet definitions

- **2*n*-pole**:

dipole quadrupole sextupole octupole …



n: 1 2 3 4 …

- Normal: gap appears at the horizontal plane
- Skew: rotate around beam axis by $\pi/2n$ angle
- Symmetry: rotating around beam axis by π/n angle, the field is reversed (polarity flipped)

Equations of motion – Linear fields

- Consider *s*-dependent fields from dipoles and normal quadrupoles $B_y = B_0(s) - g(s)x\ ,\ \ B_x = -g(s)y$
- The total momentum can be written $P = P_0(1 + \frac{\Delta P}{P})$
- With magnetic rigidity $B_0\rho = \frac{P_0}{q}$ and normalized gradient

$k(s) = \frac{g(s)}{B_0\rho}$ the equations of motion are

$$\boxed{\begin{aligned} x'' - \left(k(s) - \frac{1}{\rho(s)^2}\right) x &= \frac{1}{\rho(s)} \frac{\Delta P}{P} \\ y'' + k(s)\ y &= 0 \end{aligned}}$$

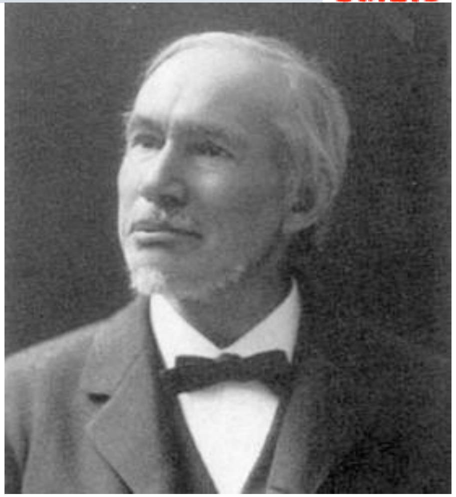
- Inhomogeneous equations with *s*-dependent coefficients
- Note that the term $1/\rho^2$ corresponds to the dipole **week focusing**
- The term $\Delta P/(P\rho)$ represents **off-momentum** particles

Hill's equations

- Solutions are combination of the ones from the homogeneous and inhomogeneous equations
- Consider particles with the design momentum. The equations of motion become

$$\boxed{\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}}$$

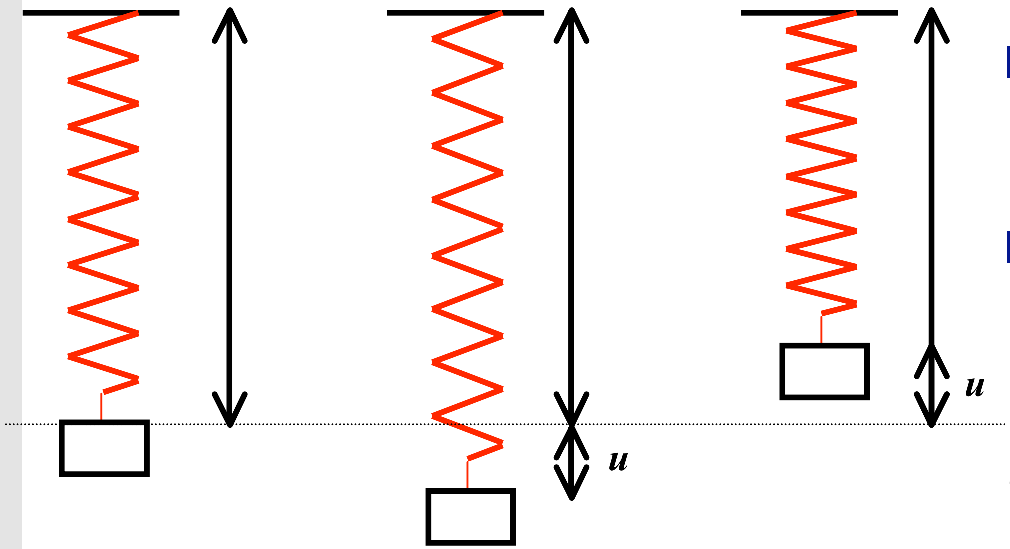
with $K_x(s) = -\left(k(s) - \frac{1}{\rho(s)^2}\right) \ , \quad K_y(s) = k(s)$



George Hill

- **Hill's equations of linear transverse particle motion**
- Linear equations with s-dependent coefficients (harmonic oscillator with time dependent frequency)
- In a ring (or in transport line with symmetries), coefficients are periodic $K_x(s) = K_x(s+C) \ , \quad K_y(s) = K_y(s+C)$
- Not straightforward to derive analytical solutions for whole accelerator

Harmonic oscillator – spring



- Consider $K(s) = k_0 = \text{constant}$

$$u'' + k_0\, u = 0$$

- Equations of harmonic oscillator with solution

$$\begin{aligned} u(s) &= C(s)\, u(0) + S(s)\, u'(0) \\ u'(s) &= C'(s) u(0) + S'(s) u'(0) \end{aligned}$$

with

$$C(s) = \cos(\sqrt{k_0} s)\ , \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0} s) \qquad \text{for } k_0 > 0$$

$$C(s) = \cosh(\sqrt{|k_0|} s)\ , \quad S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|} s) \qquad \text{for } k_0 < 0$$

- Note that the solution can be written in **matrix** form

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

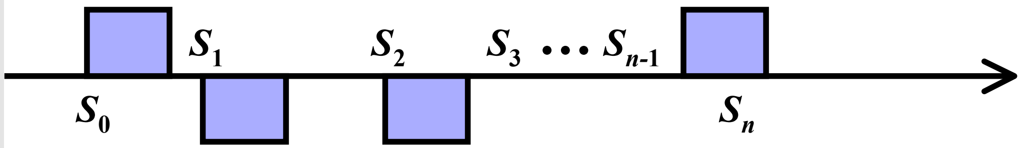
Matrix formalism

- General **transfer matrix** from s_0 to s

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

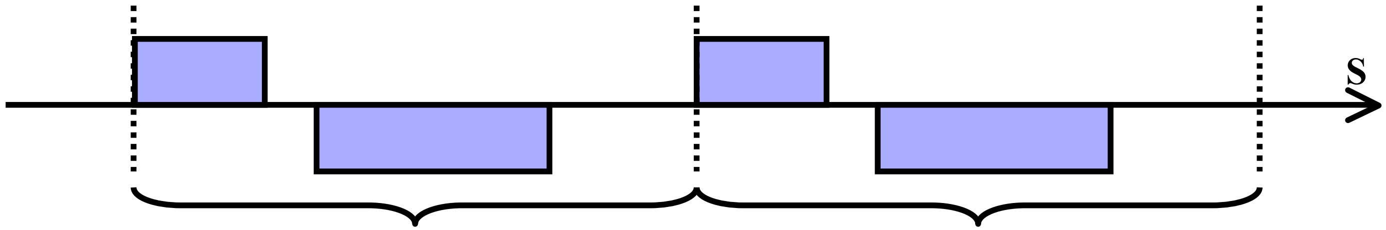
- Note that $\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$ which is always true for conservative systems
- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$
- The accelerator can be build by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \underbrace{\mathcal{M}(s_n|s_{n-1}) \ldots \underbrace{\mathcal{M}(s_3|s_2) \cdot \underbrace{\mathcal{M}(s_2|s_1) \cdot \underbrace{\mathcal{M}(s_1|s_0)}_{\text{from } \boldsymbol{s_0} \text{ to } \boldsymbol{s_1}}}_{\text{from } \boldsymbol{s_0} \text{ to } \boldsymbol{s_2}}}_{\text{from } \boldsymbol{s_0} \text{ to } \boldsymbol{s_3}}}_{\text{from } \boldsymbol{s_0} \text{ to } \boldsymbol{s_n}}$$



Symmetric lines

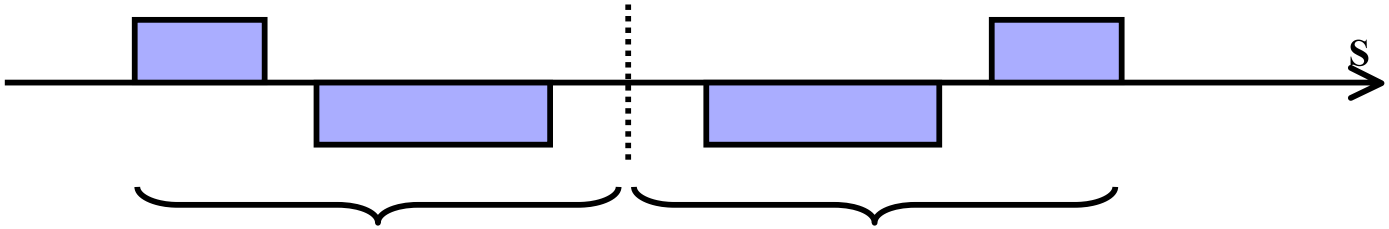
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$

4x4 Matrices

- Combine the matrices for each plane

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

Uncoupled motion

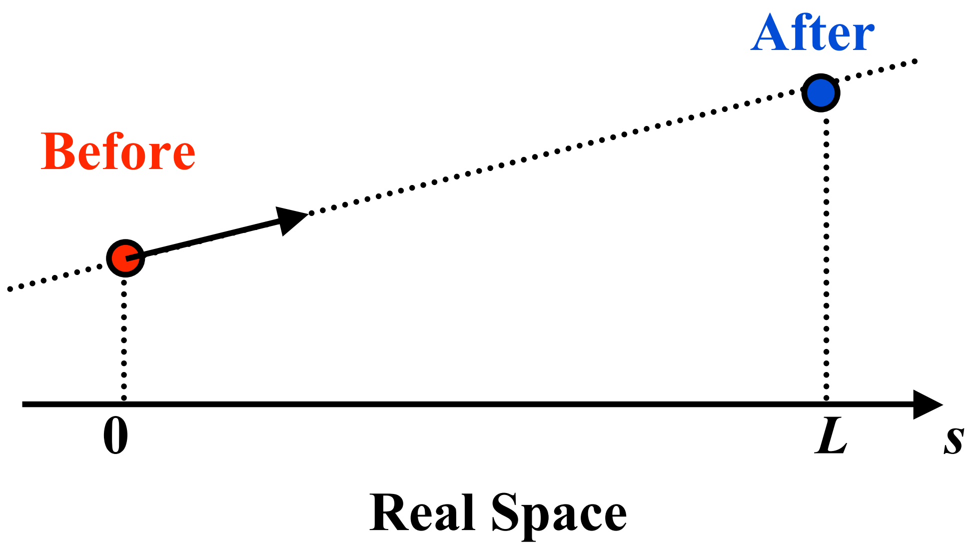
- Consider a drift (no magnetic elements) of length L=s-s_0

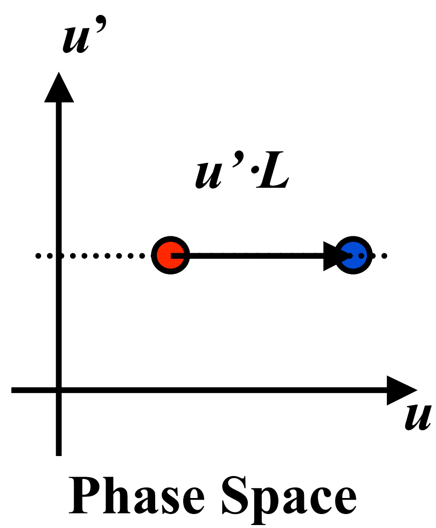
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\mathrm{drift}}(s|s_0) = \begin{pmatrix} 1 & s-s_0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{aligned} u(s) &= u_0 + \overbrace{(s-s_0)}^{L} u'_0 = u_0 + L u'_0 \\ u'(s) &= u'_0 \end{aligned}$$

- Position changes if particle has a slope which remains unchanged.





(De)focusing thin lens

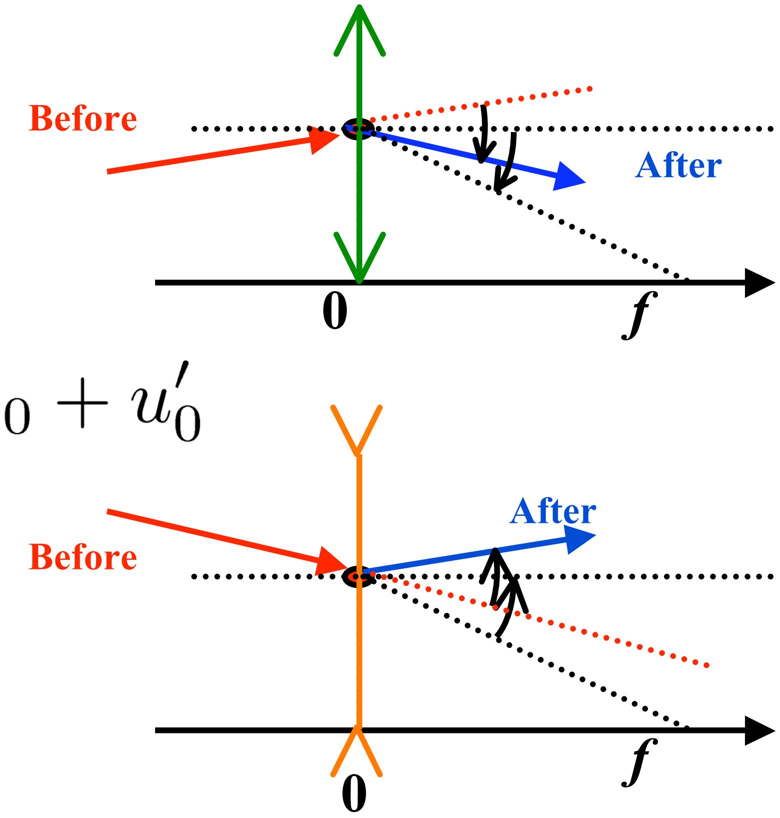
- Consider a lens with focal length $\pm f$

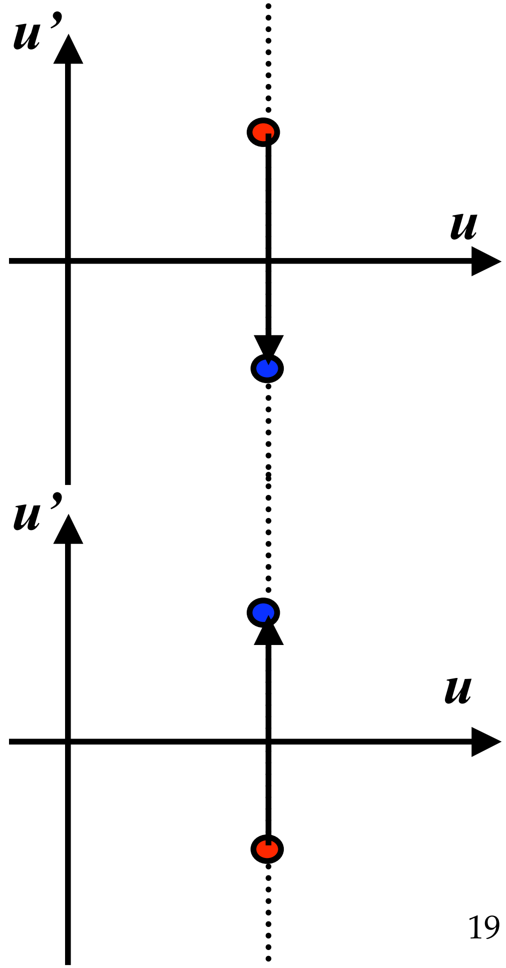
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\text{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}}$$

- Slope **diminishes** (focusing) or **increases** (defocusing) for positive position, which remains unchanged.

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= \mp\frac{1}{f}u_0 + u'_0 \end{aligned}$$

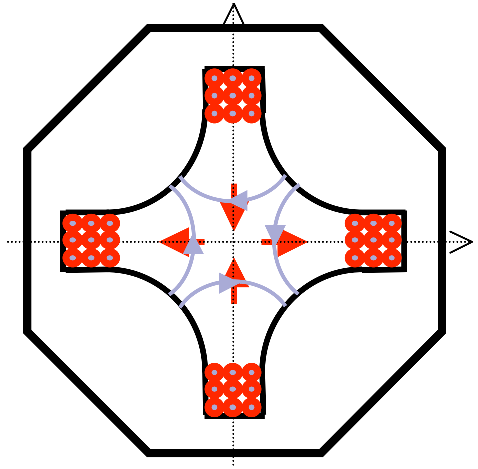




Quadrupole

- Consider a quadrupole magnet of length $L = s\text{-}s_0$. The field is

$$B_y = -g(s)x\ ,\quad B_x = -g(s)y$$

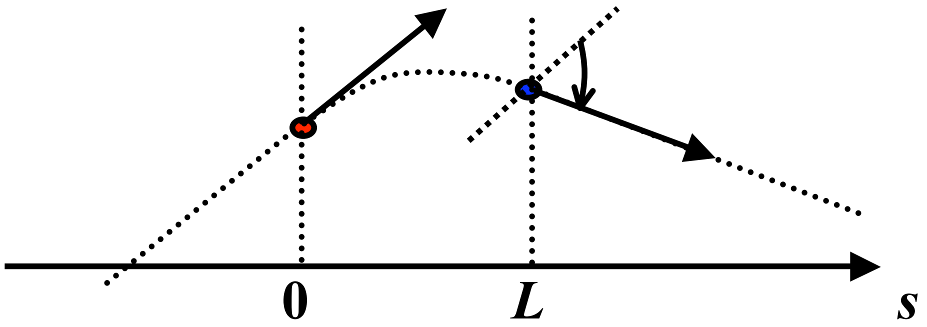


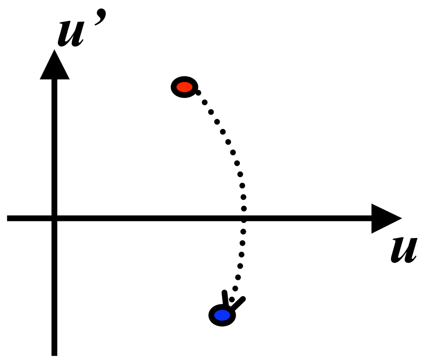
- with normalized quadrupole gradient (in $\mathbf{m}^{-2}$)

$$k(s) = \frac{g(s)}{B_0\rho}$$

The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$





(De)focusing Quadrupoles

- For a focusing quadrupole (k>0)

$$\boxed{\mathcal{M}_{\mathrm{QF}} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}}$$

- For a defocusing quadrupole (k<0)

$$\boxed{\mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} \cosh(\sqrt{|k|}L) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}L) \\ \sqrt{|k|}\sinh(\sqrt{|k|}L) & \cosh(\sqrt{|k|}L) \end{pmatrix}}$$

- By setting $\sqrt{k}L \to 0$

$$\mathcal{M}_{\mathrm{QF,QD}} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \mathcal{M}_{\mathrm{lens}}$$

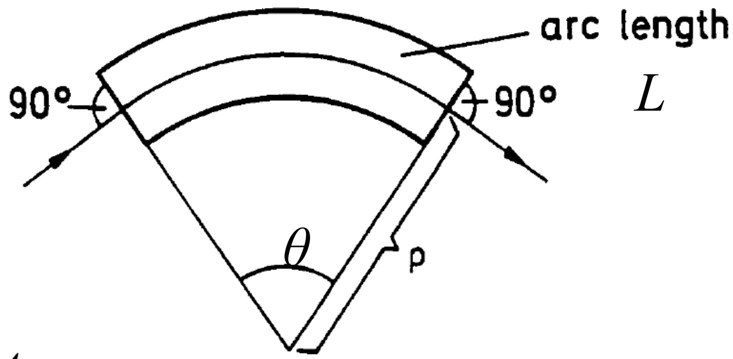
- **Note** that the **sign** of k or f is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Sector Dipole

- Consider a dipole of (arc) length L.
- By setting in the focusing quadrupole matrix $k = \frac{1}{\rho^2} > 0$ the transfer matrix for a sector dipole becomes

$$\boxed{\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}}$$

with a bending radius $\theta = \frac{L}{\rho}$



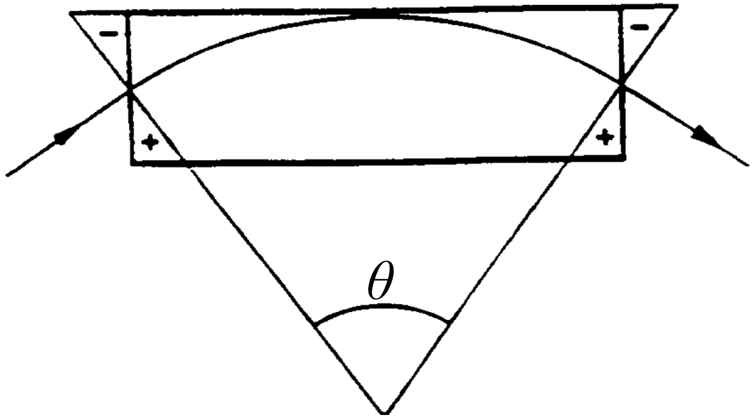
L

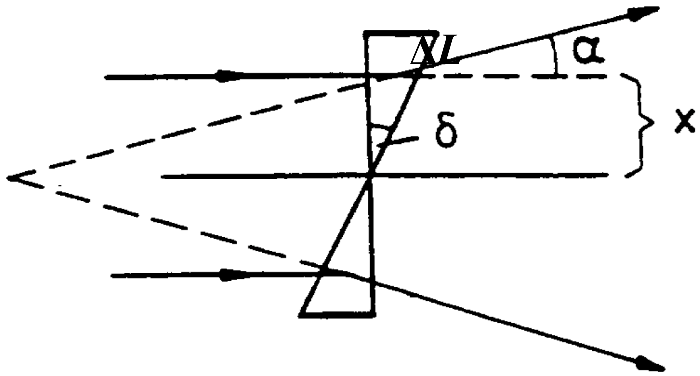
- In the non-deflecting plane $\frac{1}{\rho} \to 0$

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \mathcal{M}_{\text{drift}}$$

- This is a **hard-edge** model. In fact, there is some **edge focusing** in the vertical plane
- Matrix generalized by adding gradient (**synchrotron magnet**)

Rectangular Dipole





- Consider a rectangular dipole with bending angle θ. At each edge of length ΔL, the deflecting angle is changed by

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho}$$

i.e., it acts as a thin defocusing lens with focal length $\frac{1}{f} = \frac{\tan\delta}{\rho}$

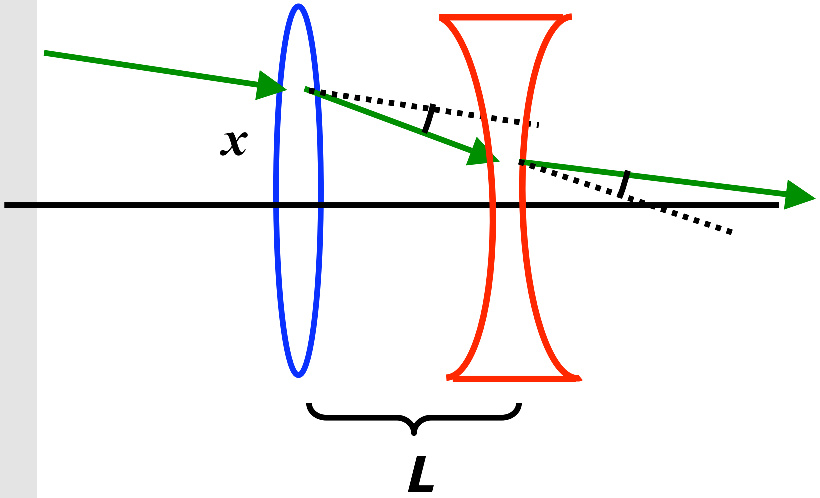
- The transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$ with

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{-\tan(\delta)}{\rho} & 1 \end{pmatrix}$$

- For **θ<<1, δ=θ/2**
- In deflecting plane (like **drift**), in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \quad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Quadrupole doublet



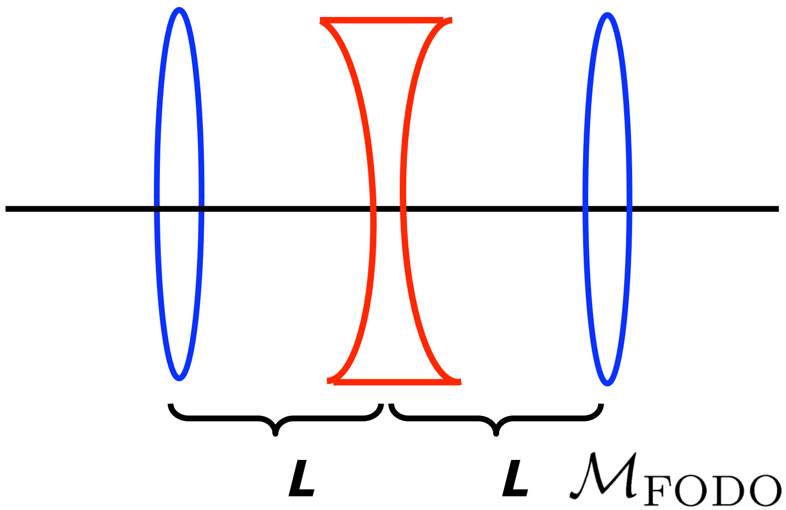
- Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths f_1 and f_2 separated by a distance L.
- In thin lens approximation the transport matrix is

$$\mathcal{M}_{\text{doublet}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^\star} & 1 - \frac{L}{f_2} \end{pmatrix}$$

with the **total focal length** $\dfrac{1}{f^\star} = \dfrac{1}{f_1} + \dfrac{1}{f_2} - \dfrac{L}{f_1 \ f_2}$

- Setting $f_1 = - f_2 = f$ $\qquad \dfrac{1}{f^\star} = \dfrac{L}{f^2}$
- **Alternating gradient focusing** seems overall focusing
- This is only valid in thin lens approximation

FODO Cell



- Consider defocusing quad "sandwiched" by two focusing quads with focal lengths $\pm f$.
- Symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

with the transfer matrices

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Outline - part II

- General solutions of Hill's equations
  - Floquet theory
- Betatron functions
- Transfer matrices revisited
  - General and periodic cell
- General transport of betatron functions
  - Drift
  - Beam waist
- Normalized coordinates
- Off-momentum particles
  - Effect from dipoles and quadrupoles
  - Dispersion equation
  - 3x3 transfer matrices
- Periodic lattices in circular accelerators
  - Periodic solutions for beta function and dispersion
  - Symmetric solution
- Tune and Working point
- Matching the optics

Solution of Betatron equations

- Betatron equations are linear

$$\begin{aligned} x'' + K_x(s)\ x &= 0 \\ y'' + K_y(s)\ y &= 0 \end{aligned}$$

with periodic coefficients

$$K_x(s) = K_x(s+C)\ , \quad K_y(s) = K_y(s+C)$$

- **Floquet theorem** states that the solutions are

$$u(s) = Aw(s)\cos(\psi(s) + \psi_0)$$

where $w(s)$, $\psi(s)$ are periodic with the same period

$$w(s) = w(s+C)\ , \quad \psi(s) = \psi(s+C)$$

- Note that solutions resemble the one of harmonic oscillator

$$u(s) = A\cos(\psi(s) + \psi_0)$$

- Substitute solution in Betatron equations

$$u'' + K(s)\ u = A\underbrace{(2w'\psi' + w\psi'')}_{0}\sin(\psi + \psi_0) + A\underbrace{(w'' - w\psi'^2 + Kw)}_{0}\cos(\psi + \psi_0) = 0$$

Betatron functions

- By multiplying with w the coefficient of sin

$$2w'w\psi' + w^2\psi'' = (w^2\psi')' = 0$$

- Integrate to get $\psi = \int \frac{ds}{w^2(s)}$

- Replace ψ' in the coefficient of cos and obtain

$$w^3(w'' + K_x w) = 1$$

- Define the **Betatron** or t**wiss** or **lattice functions** (Courant-Snyder parameters)

$$\boxed{\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \\ \gamma(s) &\equiv \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}}$$

Betatron motion

- The on-momentum linear betatron motion of a particle is described by

$$\boxed{u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \psi_0)}$$

with $\alpha,\ \beta,\ \gamma$ the twiss functions $\alpha(s) = -\frac{\beta(s)'}{2},\ \ \gamma = \frac{1+\alpha(s)^2}{\beta(s)}$

ψ the **betatron phase** $\psi(s) = \int \frac{ds}{\beta(s)}$

and the **beta function** β is defined by the **envelope equation**

$$\boxed{2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4}$$

- By differentiation, we have that the **angle** is

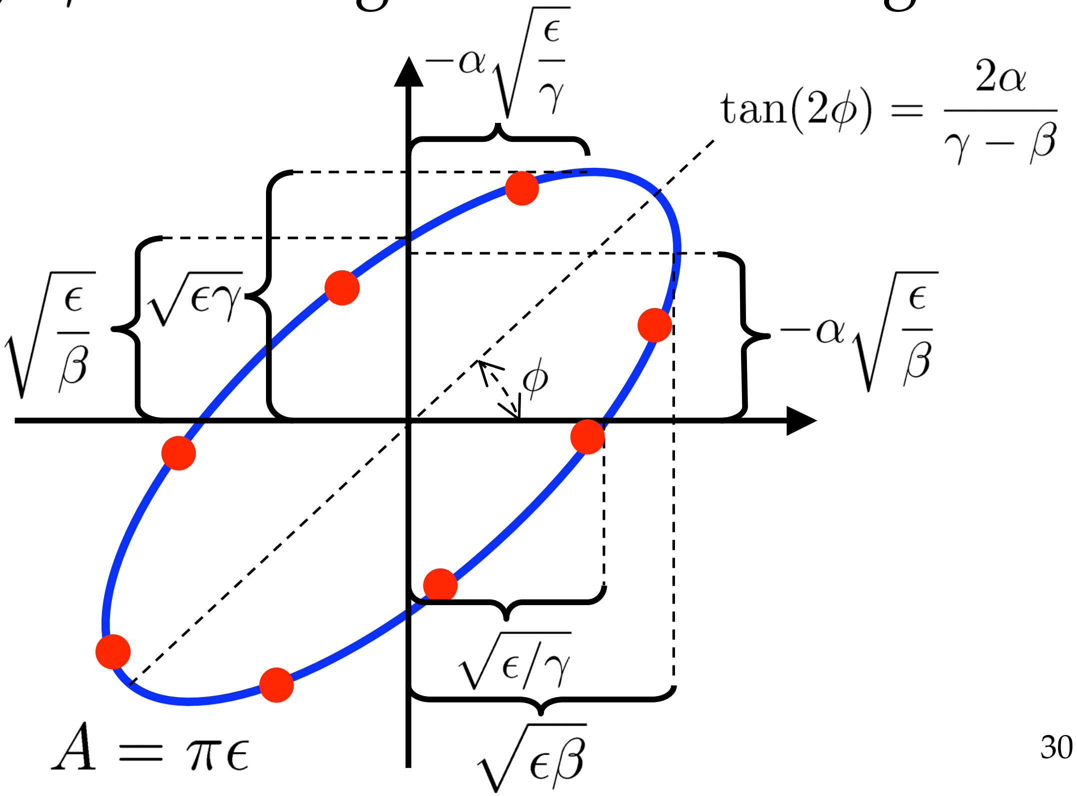
$$\boxed{u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s) + \psi_0) + \alpha(s)\cos(\psi(s) + \psi_0)\right)}$$

Courant-Snyder invariant

- Eliminating the angles by the position and slope we define the **Courant-Snyder invariant**

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \epsilon$$

- This is an ellipse in phase space with area $\pi\epsilon$
- The twiss functions $\alpha,\ \beta,\ \gamma$ have a geometric meaning



- The beam envelope is

$$E(s) = \sqrt{\epsilon\beta(s)}$$

- The beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$

General transfer matrix

- From equation for position and angle we have

$$\cos(\psi(s)+\psi_0) = \frac{u}{\sqrt{\epsilon\beta(s)}}\ , \quad \sin(\psi(s)+\psi_0) = \sqrt{\frac{\beta(s)}{\epsilon}}u' + \frac{\alpha(s)}{\sqrt{\epsilon\beta(s)}}u$$

- Expand the trigonometric formulas and set $\psi(0)=0$ to get the transfer matrix from location 0 to s

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathcal{M}_{0\to s}\begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

with

$$\mathcal{M}_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta(s)\beta_0}\sin\Delta\psi \\ \frac{(a_0-a(s))\cos\Delta\psi-(1+\alpha_0\alpha(s))\sin\Delta\psi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta_0}{\beta(s)}}(\cos\Delta\psi - \alpha_0\sin\Delta\psi) \end{pmatrix}$$

and $\Delta\psi = \int_0^s \frac{ds}{\beta(s)}$ the **phase advance**

Periodic transfer matrix

- Consider a periodic cell of length C
- The optics functions are $\beta_0 = \beta(C) = \beta \;, \quad \alpha_0 = \alpha(C) = \alpha$

  and the phase advance $$\mu = \int_0^C \frac{ds}{\beta(s)}$$

- The transfer matrix is

$$\mathcal{M}_C = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- The cell matrix can be also written as

$$\mathcal{M}_C = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the **Twiss matrix** $\boxed{\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}$

Stability conditions

- From the periodic transport matrix $\mathrm{Trace}(\mathcal{M}_C) = 2\cos\mu$ and the following stability criterion

$$\boxed{|\mathrm{Trace}(\mathcal{M}_C)| < 2}$$

- In a ring, the **tune** is defined from the 1-turn phase advance

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

i.e. number betatron oscillations per turn

- From transfer matrix for a cell we get

$$\mathcal{M}_C = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\cos\mu = \frac{1}{2}(m_{11} + m_{22}), \;\; \beta = \frac{m_{12}}{\sin\mu}, \;\; \alpha = \frac{m_{11} - m_{22}}{2\sin\mu}, \;\; \gamma = -\frac{m_{21}}{\sin\mu}$$

Transport of Betatron functions

- For a general matrix between position 1 and 2

$$\mathcal{M}_{s_1\to s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{and the inverse } \mathcal{M}_{s_2\to s_1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

- Equating the invariant at the two locations

$$\epsilon = \gamma_{s_2} {u_{s_2}}^2 + 2\alpha_{s_2} u_{s_2} u'_{s_2} + \beta_{s_2} u'^2_{s_2} = \gamma_{s_1} {u_{s_1}}^2 + 2\alpha_{s_1} u_{s_1} u'_{s_1} + \beta_{s_1} u'^2_{s_1}$$

and eliminating the transverse positions and angles

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & -m_{22}m_{12} \\ m_{21}^2 & 2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

Example I: Drift

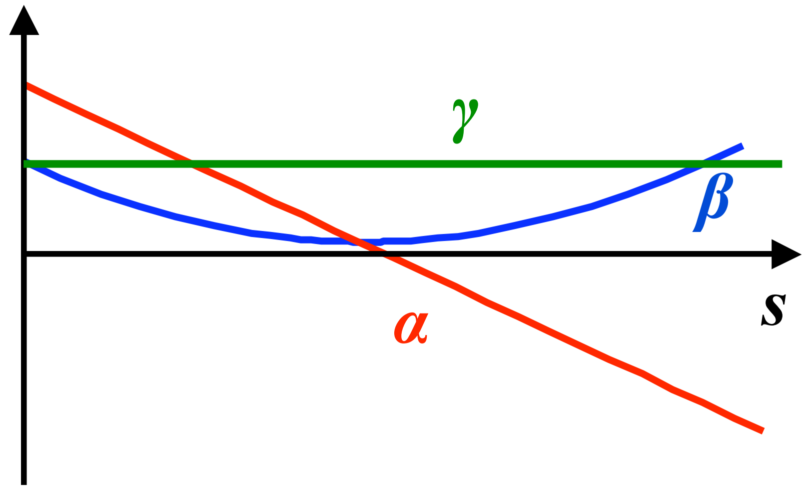
- Consider a drift with length s

- The transfer matrix is $$\mathcal{M}_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

- The betatron transport matrix is $$\begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$$

from which

$$\begin{aligned} \beta(s) &= \beta_0 - 2s\alpha_0 + s^2\gamma_0 \\ \alpha(s) &= \alpha_0 - s\gamma_0 \\ \gamma(s) &= \gamma_0 \end{aligned}$$



Simplified method for betatron transport

- Consider the beta matrix $\mathcal{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ the matrix

$\mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and its transpose $\mathcal{M}^T_{1\to 2} = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$

- It can be shown that

$$\boxed{\mathcal{B}_2 = \mathcal{M}_{1\to 2} \cdot \mathcal{B}_1 \cdot \mathcal{M}^T_{1\to 2}}$$

- Application in the case of the drift

$$\mathcal{B} = \mathcal{M}_{\text{drift}} \cdot \mathcal{B}_0 \cdot \mathcal{M}^T_{\text{drift}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

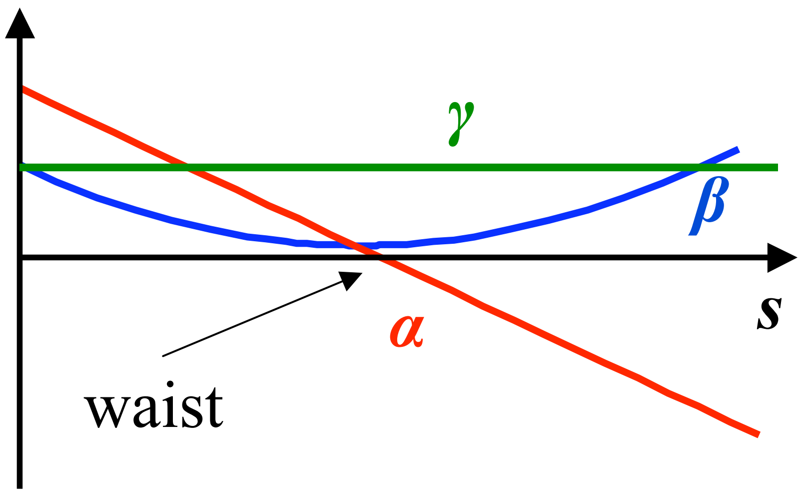
and

$$\mathcal{B} = \begin{pmatrix} \beta_0 - 2s\alpha_0 + s^2\gamma_0 & -\alpha_0 + s\gamma_0 \\ -\alpha_0 + s\gamma_0 & \gamma_0 \end{pmatrix}$$

Example II: Beam waist

- For beam waist α=0 and occurs at s = α_0 / γ_0
- Beta function grows quadratically and is minimum in waist

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$



- The beta at the waste for having beta minimum $\frac{d\beta(s)}{d\beta_0} = 0$

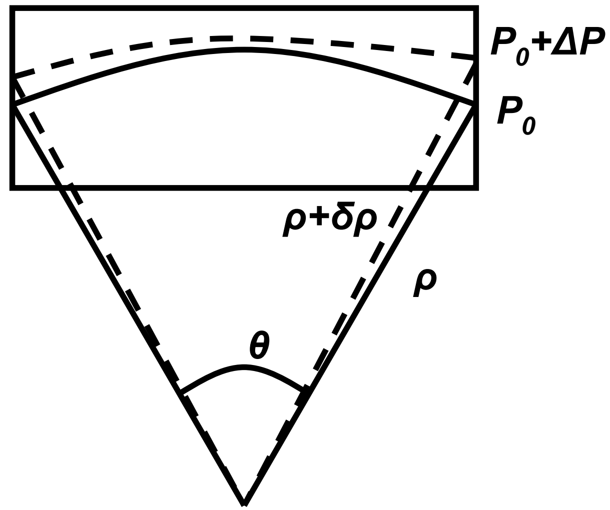
in the middle of a drift with length L is $\beta_0 = \frac{L}{2}$

- The phase advance of a drift is $\mu = \int_0^{L/2} \frac{ds}{\beta(s)} = \arctan(\frac{L}{2\beta_0})$

which is π/2 when $\beta_0 \to \infty$. Thus, for a drift $\boxed{\mu = \leq \pi}$

Effect of dipole on off-momentum particles

- Up to now all particles had the same momentum P_0
- What happens for off-momentum particles, i.e. particles with momentum $P_0+\Delta P$?
- Consider a dipole with field B and bending radius ρ
- Recall that the magnetic rigidity $B\rho = \frac{P_0}{q}$ and for off-momentum particles



$$B(\rho + \Delta\rho) = \frac{P_0 + \Delta P}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P_0}$$

- Considering the effective length of the dipole unchanged

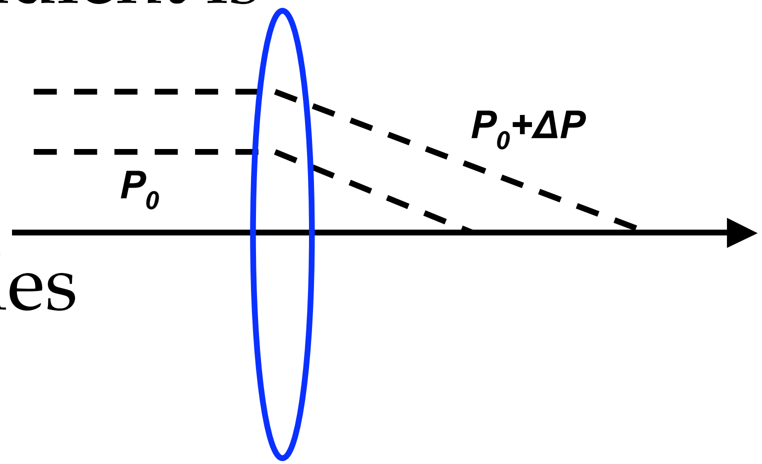
$$\theta\rho = l_{eff} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta P}{P_0}$$

- Off-momentum particles get different deflection (different orbit)

$$\boxed{\Delta\theta = -\theta\frac{\Delta P}{P_0}}$$

Off-momentum particles and quadrupoles

- Consider a quadrupole with gradient ***G***
- Recall that the normalized gradient is



$$K = \frac{q\,G}{P_0}$$

and for off-momentum particles

$$\Delta K = \frac{dK}{dP}\Delta P = -\frac{qG}{P_0}\frac{\Delta P}{P_0}$$

- Off-momentum particle gets different focusing

$$\Delta K = -K\frac{\Delta P}{P_0}$$

- This is equivalent to the effect of **optical lenses** on **light of different wavelengths**

Dispersion equation

- Consider the equations of motion for off-momentum particles

$$x'' + K_x(s)x = \frac{1}{\rho(s)}\frac{\Delta P}{P}$$

- The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum)

$$x(s) = x_H(s) + x_I(s)$$

- In that way, the equations of motion are split in two parts

$$\begin{aligned} x_H'' + K_x(s)x_H &= 0 \\ x_I'' + K_x(s)x_I &= \frac{1}{\rho(s)}\frac{\Delta P}{P} \end{aligned}$$

- The **dispersion function** can be defined $D(s) = \dfrac{x_I(s)}{\Delta P/P}$
- The dispersion equation is

$$\boxed{D''(s) + K_x(s)\ D(s) = \frac{1}{\rho(s)}}$$

Dispersion solution for a bend

- Simple solution by considering motion through a sector dipole with constant bending radius ρ

- The dispersion equation becomes $$D''(s) + \frac{1}{\rho^2}D(s) = \frac{1}{\rho}$$

- The solution of the homogeneous is harmonic with frequency $1/\rho$
- A particular solution for the inhomogeneous is $D_p = \text{constant}$ and we get by replacing $D_p = \rho$
- Setting $D(0) = D_0$ *and* $D'(0) = D_0'$, the solutions for dispersion are

$$\begin{aligned} D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0'\rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\ D'(s) &= -\frac{D_0}{\rho}\sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho}) \end{aligned}$$

General dispersion solution

- General solution possible with perturbation theory and use of Green functions
- For a general matrix $\mathcal{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S(s)' \end{pmatrix}$ the solution is

$$D(s) = S(s)\int_{s_0}^{s} \frac{C(\bar{s})}{\rho(\bar{s})} d\bar{s} + C(s)\int_{s_0}^{s} \frac{S(\bar{s})}{\rho(\bar{s})} d\bar{s}$$

- One can verify that this solution indeed satisfies the differential equation of the dispersion (and the sector bend)
- The general betatron solutions can be obtained by 3X3 transfer matrices including dispersion

$$\mathcal{M}_{3\times3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

- Recalling that $x(s) = x_B(s) + D(s)\dfrac{\Delta P}{P}$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \mathcal{M}_{3\times3} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

- For **drifts** and **quadrupoles** which do not create dispersion the 3x3 transfer matrices are just

$$\mathcal{M}_{\text{drift,quad}} = \begin{pmatrix} \mathcal{M}_{2\times 2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

- For the deflecting plane of a **sector bend** we have seen that the matrix is

$$\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and in the non-deflecting plane is just a drift.

- Synchrotron magnets have focusing and bending included in their body.
- From the solution of the sector bend, by replacing *1/ρ* with

$$\sqrt{K} = \sqrt{\frac{1}{\rho^2} - k}$$

- For $K>0$ $$\mathcal{M}_{\mathrm{syF}} = \begin{pmatrix} \cos\psi & \frac{\sin\psi}{\sqrt{K}} & \frac{1-\cos\psi}{\rho K} \\ -\sqrt{K}\sin\psi & \cos\psi & \frac{\sin\psi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$$

- For $K<0$ $$\mathcal{M}_{\mathrm{syD}} = \begin{pmatrix} \cosh\psi & \frac{\sinh\psi}{\sqrt{|K|}} & -\frac{1-\cosh\psi}{\rho|K|} \\ \sqrt{|K|}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$$

with $$\psi = \sqrt{|k + \frac{1}{\rho^2}|}l$$

3x3 transfer matrices - Rectangular magnet

- The end field of a rectangular magnet is simply the one of a quadrupole. The transfer matrix for the edges is

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan(\theta/2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

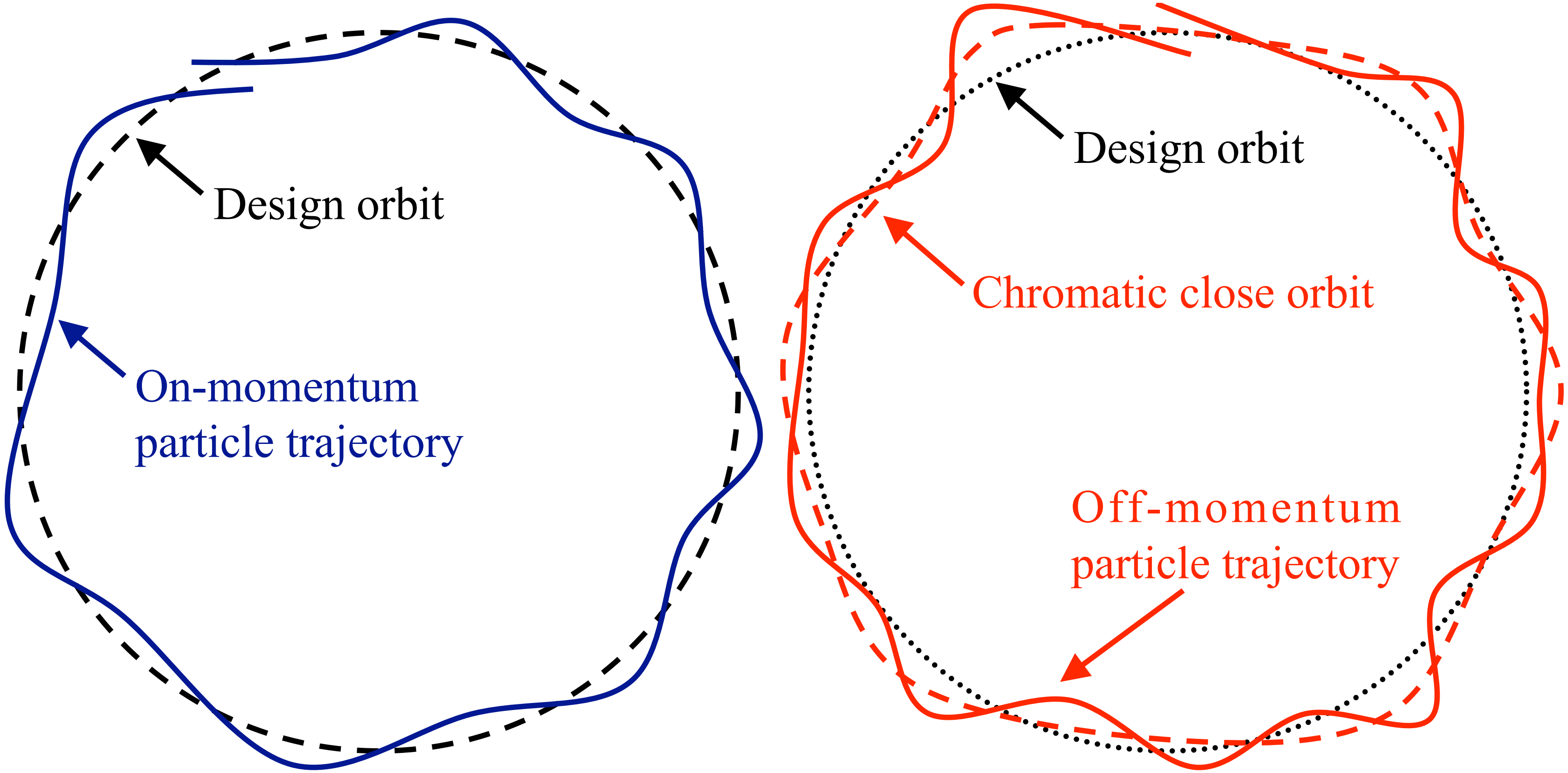
- The transfer matrix for the body of the magnet is like for the sector bend $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sect}} \cdot \mathcal{M}_{\text{edge}}$
- The total transfer matrix is

$$\mathcal{M}_{\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan(\theta/2) \\ 0 & 0 & 1 \end{pmatrix}$$

Chromatic closed orbit

- Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit
- Distance from the design orbit depends linearly with momentum spread and dispersion

$$x_D = D(s)\frac{\Delta P}{P}$$



Periodic solutions

- Consider two points s_0 and s_1 for which the magnetic structure is repeated.
- The optical function follow periodicity conditions

$$\beta_0 = \beta(s_0) = \beta(s_1)\ , \quad \alpha_0 = \alpha(s_0) = \alpha(s_1)$$
$$D_0 = D(s_0) = D(s_1)\ , \ D_0' = D'(s_0) = D'(s_1)$$

- The beta matrix at this point is $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix}$
- Consider the transfer matrix from s_0 to s_1 $\quad \mathcal{M}_{1\to 2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

$$\mathcal{B}_0 = \mathcal{M}_{0\to 1} \cdot \mathcal{B}_0 \cdot \mathcal{M}_{0\to 1}^T \Rightarrow \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \frac{2m_{12}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$
$$\alpha_0 = \frac{m_{11} - m_{22}}{\sqrt{2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2}}$$

with the condition $2 - m_{11}^2 - 2m_{12}m_{21} - m_{22}^2 > 0$

Periodic solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

- Solve for the dispersion and its derivative to get

$$D_0' = \frac{m_{21}m_{13} + m_{23}(1 - m_{11})}{2 - m_{11} - m_{22}}$$

$$D_0 = \frac{m_{12}D_0' + m_{13}}{1 - m_{11}}$$

with the conditions $m_{11} + m_{22} \neq 2$ and $m_{11} \neq 1$

Symmetric solutions

- Consider two points s_0 and s_1 for which the lattice is mirror symmetric
- The optical function follow periodicity conditions

$$\alpha(s_0) = \alpha(s_1) = 0$$
$$D'(s_0) = D'(s_1) = 0$$

- The beta matrices at s_0 and s_1 are $\mathcal{B}_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$ $\mathcal{B}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$
- Considering the transfer matrix between s_0 and s_1

$$\mathcal{B}_1 = \mathcal{M}_{0\to1} \cdot \mathcal{B}_0 \cdot \mathcal{M}_{0\to1}^T \Rightarrow \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- The solution for the optics functions is

$$\beta_0 = \sqrt{-\frac{m_{12}m_{22}}{m_{21}m_{11}}} \quad \text{and} \quad \beta_1 = -\frac{1}{\beta_0}\frac{m_{12}}{m_{21}}$$

with the condition $\frac{m_{12}}{m_{21}} < 0$ and $\frac{m_{22}}{m_{11}} > 0$

Symmetric solutions for dispersion

- Consider the 3x3 matrix for propagating dispersion between s_0 and s_1

$$\begin{pmatrix} D(s_1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_0) \\ 0 \\ 1 \end{pmatrix}$$

- Solve for the dispersion in the two locations

$$\begin{aligned} D(s_0) &= -\frac{m_{23}}{m_{21}} \\ D(s_1) &= -\frac{m_{11}m_{23}}{m_{21}} + m_{13} \end{aligned}$$

- Imposing certain values for beta and dispersion, quadrupoles can be adjusted in order to get a solution

- Consider a general periodic structure of length $2L$ which contains **N** cells. The transfer matrix can be written as

$$\mathcal{M}(s+N\cdot 2L|s) = \mathcal{M}(s+2L|s)^N$$

- The periodic structure can be expressed as

$$\mathcal{M} = \mathcal{I}\cos\mu + \mathcal{J}\sin\mu$$

with $\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$.

- Note that because $\det(\mathcal{M}) = 1 \rightarrow \beta\gamma - a^2 = 1$
- Note also that $\mathcal{J}^2 = -\mathcal{I}$
- By using **de Moivre's formula**

$$\mathcal{M}^N = (\mathcal{I}\cos\mu + \mathcal{J}\sin\mu)^N = \mathcal{I}\cos(N\mu) + \mathcal{J}\sin(N\mu)$$

- We have the following general stability criterion

$$|\mathrm{Trace}(\mathcal{M}^N)| = 2\cos(N\mu) < 2$$

3X3 FODO cell matrix

- Insert a sector dipole in between the quads and consider $\theta = L/\rho << 1$
- Now the transfer matrix is $\mathcal{M}_{\mathrm{HFODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{sector}} \cdot \mathcal{M}_{\mathrm{HQD}}$ which gives

$$\mathcal{M}_{\mathrm{HFODO}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{L^2}{2\rho} \\ 0 & 1 & \frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and after multiplication

$$\mathcal{M}_{\mathrm{HFODO}} = \begin{pmatrix} 1 - \frac{L}{f} & L & \frac{L^2}{(2\rho)} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} & \frac{L}{\rho}(1 + \frac{L}{2f}) \\ 0 & 0 & 1 \end{pmatrix}$$

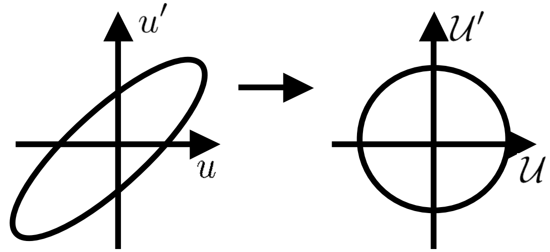
General solution for the dispersion

- Introduce **Floquet variables**

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\ ,\ \ \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\ ,\ \ \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int \frac{ds}{\beta(s)}$$

- The Hill's equations are written $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$
- The solutions are the ones of an harmonic oscillator

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix} \cos(\nu\phi) \\ -\sin(\nu\phi) \end{pmatrix}$$



- For the dispersion solution $\mathcal{U} = \frac{D}{\sqrt{\beta}}\frac{\Delta P}{P}$, the inhomogeneous equation in Floquet variables is written

$$\frac{d^2 D}{d\phi^2} + \nu^2 D = -\frac{\nu^2\beta^{3/2}}{\rho(s)}$$

- This is a forced harmonic oscillator with solution

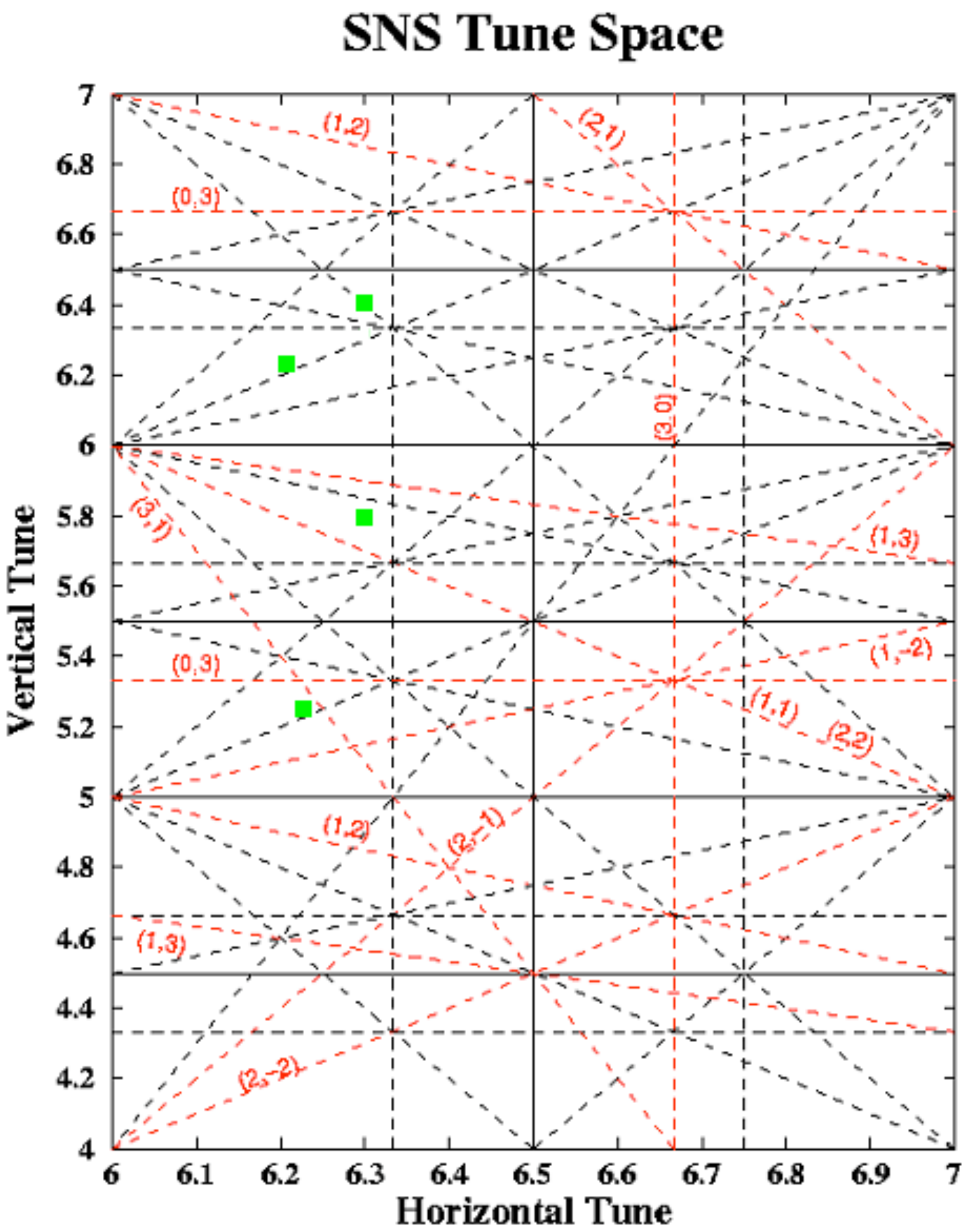
$$D(s) = \frac{\sqrt{\beta(s)}\nu}{2\sin(\pi\nu)}\oint \frac{\sqrt{\beta(\sigma)}}{\rho(\sigma)}\cos[\nu(\phi(s) - \phi(\sigma) + \pi)]d\sigma$$

- Note the **resonance conditions** for integer tunes!!!

Tune and working point

- In a ring, the **tune** is defined from the 1-turn phase advance
$$Q_{x,y} = \frac{1}{2\pi} \oint \frac{ds}{\beta_{x,y}(s)}$$
i.e. number betatron oscillations per turn
- Taking the average of the betatron tune around the ring we have in **smooth approximation**
$$2\pi Q = \frac{C}{\langle\beta\rangle} \rightarrow Q = \frac{R}{\langle\beta\rangle}$$
- Extremely useful formula for deriving scaling laws
- The position of the tunes in a diagram of horizontal versus vertical tune is called a **working point**
- The tunes are imposed by the choice of the quadrupole strengths
- One should try to avoid **resonance conditions**

Example: SNS Ring Tune Space



Tunability: 1 unit in horizontal, 3 units in vertical (2 units due to bump/chicane perturbation)

– **Structural resonances (up to 4th order)**

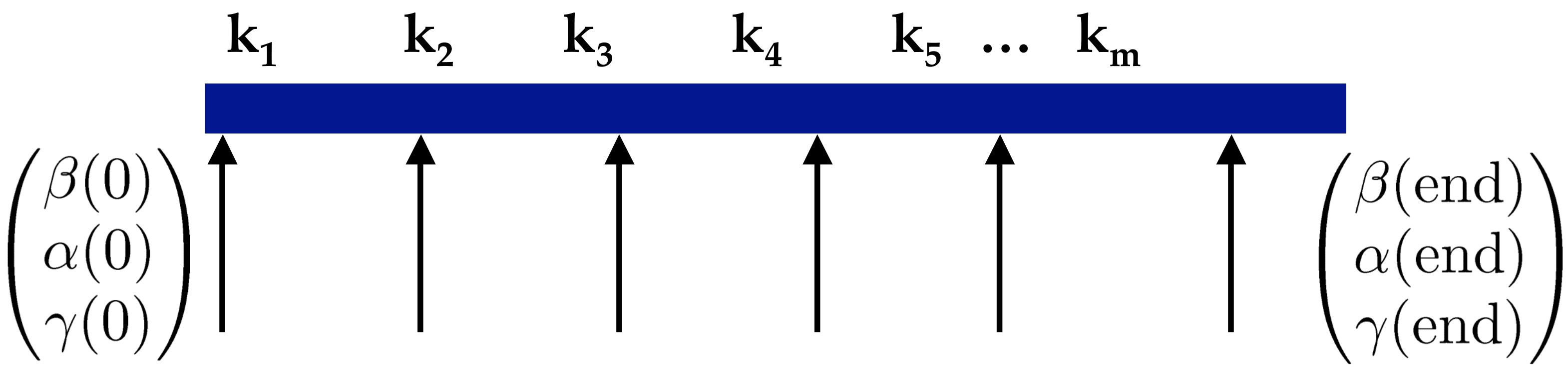
– **All other resonances (up to 3rd order)**

- **Working points considered**
  - **(6.30,5.80) - Old**
  - **(6.23,5.24)**
  - **(6.23,6.20) - Nominal**
  - **(6.40,6.30) - Alternative**

Matching the optics

- Optical function at the **entrance** and **end** of accelerator may be fixed (pre-injector, or experiment upstream)
- Evolution of optical functions determined by magnets through transport matrices
- Requirements for aperture constrain optics functions all along the accelerator
- The procedure for choosing the quadrupole strengths in order to achieve all optics function constraints is called **matching of beam optics**
- Solution is given by numerical simulations with dedicated programs (MAD, TRANSPORT, SAD, BETA, BEAMOPTICS) through multi-variable minimization algorithms

magnet structure



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